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BULLETIN OF THE AMERICAN MATHEMATICAL SOCIETY

A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE

EDITED BY

EARLE RAYMOND HEDRICK
WALLIE ABRAHAM HURWITZ JOHN WESLEY YOUNG

WITH THE ASSISTANCE OF

ROBERT WILBUR BURGESS DERRICK NORMAN LEHMER
DUNHAM JACKSON TULLIO LEVI-CIVITA
EDWARD KASNER HENRY LEWIS RIETZ

VOLUME XXIX

JANUARY TO DECEMBER, 1923

PUBLISHED BY THE SOCIETY
LANCASTER, PA., AND NEW YORK
1923

PRESS OF
THE NEW ERA PRINTING COMPANY
LANCASTER, PA.

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BULLETIN
OF THE
AMERICAN MATHEMATICAL SOCIETY

THE OCTOBER MEETING OF THE SOCIETY

The two hundred twenty-fourth regular meeting of the Society was held at Columbia University on Saturday, October 28, 1922, extending through the usual morning and afternoon sessions. The attendance included the following forty-eight members:

Alexander, Archibald, Barnum, Borden, Bowden, B. H. Camp, Cole, Cowley, Crum, L. D. Cummings, Douglas, Eisenhart, Fenn, Fields, Fine, Fite, Gill, Gronwall, Haskins, Hazlett, Hebbert, Hill, Hille, Joffe, Kasner, Lamond, Lamson, McDonnell, MacDuffee, MacNeish, Mathews, Mullins, Northcott, Pell, Pfeiffer, Reddick, R. G. D. Richardson, Ritt, Ruger, Seely, D. E. Smith, Sosnow, H. D. Thompson, Veblen, Weisner, H. S. White, Whited, Whittemore.

At the meeting of the Council, the following twenty-one persons were elected to membership in the Society:

Professor Margaret Buchanan, University of West Virginia;
Mr. Franz J. Feinler, Ladysmith, B. C.;
Mr. James Garfield Garrison, Polytechnic High School, San Francisco;
Mr. Charles Hopkins, University of Illinois;
Professor Daniel Hull, University of Notre Dame;
Mr. LeRoy Archibald MacColl, Western Electric Company;
Mr. Donald Hector MacPherson, Brown University;
Miss Frances Morrill Merriam, Wellesley College;
Mr. Edward Charles Molina, American Telephone and Telegraph Company;
Professor Paul Muehlman, Marquette University;
Miss Margaret Comstock Packer, Hood College;
Professor Llewellyn Rood Perkins, Middlebury College;
Mr. Emeterio Roa, University of Michigan;
Mr. William E. Roth, Phillips, Wis.;
Professor Hazel Edith Schoonmaker, Western College for Women;
Dr. Walter Andrew Shewhart, Western Electric Company;
Dr. Harry Melvin Shoemaker, North East High School, Philadelphia;
Professor John Theobald, Columbia College, Dubuque, Ia.;
Professor Clarence Eugene Van Horn, Judson College, Rangoon, Burma;
Mr. Wesley John Wagner, Purdue University;
Mr. Raymond Louis Wilder, University of Texas.

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Twenty-two applications for membership in the Society were received.

A committee consisting of Mr. S. A. Joffe and Professor W. J. Berry was appointed to audit the accounts of the Treasurer for the current year, and to report to the Council at the Annual Meeting.

A list of nominations for officers and other members of the Council was presented by the Committee on Nominations, and was unanimously adopted by the Council At the head of this list was the nomination of Professor F. N. Cole for President of the Society. The Secretary reported that Professor Cole, while appreciating the honor done him, found himself unable to accept the nomination. The Council with regret accepted this decision, and adopted an alternative nomination presented by the Committee. The following resolution was adopted:

We, the Council of the American Mathematical Society, desire to place on record an expression of our profound regret that Professor Cole feels compelled to decline the nomination to the presidency of the Society. We believe that the members of the Society in general will share our disappointment that the opportunity is thus denied us to confer on Professor Cole the honor which would most suitably express our high esteem of him and of his signal services to the Society.

The Committee on the Cole Fund presented a report recommending that the fund be used to endow a prize to be called the Frank Nelson Cole Prize in Algebra. The recommendations, which are printed elsewhere in this BULLETIN, were accepted by the Council.

Professor J. K. Whittemore presided at the morning session of the Society, relieved in the afternoon by Professor C. N. Haskins. Titles and abstracts of the papers read at this meeting follow below. Mr. Linfield was introduced by Professor Birkhoff. The papers of Professor Jackson, Mr. Linfield, Mr. Rice, and Dr. Post, and Dr. Hille's third paper were read by title.

1. Professor Edward Kasner: *Parallels and geodesics in Weyl's affine geometry.*

The author first shows that the law of parallelism in a Weyl affine-connected manifold determines the geodesics (paths), but that the converse is not true. For the case $n = 4$, for example, 40 functions $\Gamma^{\alpha}_{\beta\gamma}$ (parallelism coefficients) determine the parallels, but 36 combinations (geodesic coefficients) determine the geodesics. Two sets of coefficients $\Gamma^{\alpha}_{\beta\gamma}$ have the same geodesics when they differ by $\frac{1}{2}(\delta_{\beta}{}^{\alpha}\, l_{\gamma} + \delta_{\gamma}{}^{\alpha}\, l_{\beta})$, where $\delta_{\beta}{}^{\alpha} = 1$ or 0 as $\alpha = \beta$ or $\alpha \neq \beta$, and l_1, l_2, l_3, l_4 are arbitrary functions. It is then shown if the geodesics in any Weyl manifold are represented, by means of an arbitrary point-to-point representation, on a euclidean space, they have a certain simple geometric property, called the *cubic property*, and that this is entirely characteristic. For the case of two dimensions the result is that the locus of the centers of curvature of the curves through a given point is a special cubic curve as stated in AMERICAN JOURNAL OF MATHEMATICS, vol. 28 (1906), pp. 207, 208. The extension to n dimensions now given is immediate, and, as remarked, is sufficient to characterize a Weyl family of geodesics or paths.

2. Professor Edward Kasner: *Einstein's equations of the second and third kinds.*

The three kinds of gravitational equations were introduced by Einstein in 1915, 1917, 1919 respectively. In the general form where the energy tensor $T_{\alpha\beta}$ appears in the right-hand member each kind is an actual extension of the previous kind. The present author gives a simple proof that if this tensor vanishes (space free from matter) the equations of second and third kinds are mathematically equivalent. An incidental result is that if in the cosmological equations $R_{\alpha\beta} - \lambda g_{\alpha\beta} = 0$ (sometimes called the De Sitter equations) we do not *assume* that λ is a constant, but merely assume that λ is a point-function, we can *prove* from the system itself that λ must be a constant. Thus we have a kind of analogue of Schur's famous theorem on spaces of constant Riemann curvature.

3. Professor Oswald Veblen: *Projective and affine geometry of paths.*

This paper contains a proof that any two affine geometries within the same projective geometry of paths are related by means of a vector and, conversely, that any affine geometry and a covariant vector determine another affine geometry. It will be published in the PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES.

4. Dr. G. A. Pfeiffer: *Theorems on irreducible continua.*

The following theorems are proved in this paper: (I) If M is a bounded continuum which is irreducible between two points and C is a proper subcontinuum of M which contains one of these points, then $M - C$ is connected. (II) If M and N are bounded continua which are irreducible between the pairs of points a and b and b and c respectively, and if $M + N$ is irreducible between a and c, then any component of the product $M \cdot N$ which is not a point is either a continuum of condensation of $M + N$ or a non-decomposable continuum. (III) If M and N are bounded continua which are both irreducible between the same pair of points, then $M + N$ is a continuum which is not irreducible between any pair of points. (IV) If M is a continuum which is irreducible between the points a and b and C is a continuum of condensation of M which contains b, then M is irreducible between a and any point of C.

5. Dr. G. A. Pfeiffer: *On the mapping of dyadic sets.*

The writer shows that if A and B are two dyadic* sets and if A', a closed subset of A which is nowhere dense in A, and B', a closed subset of B which is nowhere dense in B, are homeomorphic, then there exists a (1–1) continuous correspondence between A and B which on A' is identical with any given (1–1) continuous correspondence between A' and B'. Some conditions on A' and B' other than mere homeomorphism between them is necessary for a theorem like the above.

6. Dr. Jesse Douglas: *On the analysis situs of the plane when the (directed) line is taken as element.*

The author has previously made use of a representation of the (directed) lines of the plane on a cylinder (this BULLETIN, vol. 28, p. 398). This correspondence being one–one and continuous, the analysis situs of the field of oriented lines of the plane is the same as that of the cylinder. There are two types of simple, continuous, closed curves in the plane, not equivalent under one–one continuous line transformation,

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*See Hausdorff, *Grundzüge der Mengenlehre*, p. 322.

corresponding to curves surrounding or not surrounding the cylinder. The above is on the assumption of a metric plane. If the convention is made of a single line at infinity, the oriented lines of the plane have the analysis situs of the projective cone, the vertex corresponding to the line at infinity. If we postulate two lines at infinity, oriented clockwise and counter clockwise respectively, the analysis situs is that of the sphere. If ∞^1 lines are assumed at infinity, one parallel to each direction, the analysis situs is that of the torus.

7. Dr. Jesse Douglas: *Note on the integral of mean curvature over a surface.*

Let a polyhedron be inscribed in a portion Σ of a surface, and the number of its vertices increased indefinitely, while its faces grow smaller in such a way as to approach to tangent planes to the surface. Form the sum $\Sigma e_i A_i$, where e_i denotes any edge and A_i the infinitesimal dihedral angle of the faces meeting in that edge, A_i being counted as positive or negative according to a convention based on concavity and convexity considerations. The summation is to extend over all the edges of the polyhedron. Then the limit of the above sum is equal to

$$\int\int\left(\frac{1}{R_1}+\frac{1}{R_2}\right)d\sigma$$

taken over Σ. If the surface is minimal, this limit is equal to zero for every portion of the surface, and conversely. This suggests a possible approach to the solution of Plateau's problem, based on obtaining polyhedral approximations.

8. Professor Dunham Jackson: *Note on quartiles and allied measures.*

This paper appears in the present number of this BULLETIN.

9. Mr. B. Z. Linfield: *Particle geometry.*

After setting up a set of postulates for the general n-dimensional particle varieties, the author considers primarily the dense and normal varieties, U_n and S_n respectively. For them he proves a number of separation theorems which finally lead to the definition of the connectivity of an S_n. He concludes with a number of theorems concerning the S_2 of lowest connectivity, pointing out that the postulates for an S_2 of lowest connectivity form a complete set of axioms for the four-color problem and that thus the theorems concerning the S_2 of this connectivity are immediately transferable to theorems in the four-color problem.

10. Mr. B. Z. Linfield: *On certain polar curves with applications to the location of the zeros of the pth derivative of a rational function.*

The process of polarization is here used on rational fractional functions of three homogeneous variables. After some of the properties of these curves are established the results are applied to the generalized Van den Berg curves,

$$\partial^r (ux_i + vy_i + w)^{\mu_i} / \partial w^r = 0.$$

A further study yields results about the zeros of the pth derivative of a rational function from which the root-polygon theorem follows as a particular case.

11. Mr. L. H. Rice: *On the expression of the sum of any two determinants as a determinant of more dimensions.*

In this paper it is shown that the sum of any two determinants A and B of the same number of dimensions, the same signancy, and the same order, can be expressed as a determinant C of the next higher class (number of dimensions). The sum is so formed that the elements of C are of a very simple character, being in fact identical with individual elements of A and B. It appears that upon changing the sign of every element of a certain sublayer of C, the signancy of C may be varied. This paper appears in the JOURNAL OF MATHEMATICS AND PHYSICS OF THE MASSACHUSETTS INSTITUTE OF TECHNOLOGY, vol. 1, No. 3.

12. Dr. Einar Hille: *A Pythagorean functional equation.*

The author proves that the general solution of the functional equation

$$|f(z)|^2 = |f(x)|^2 + |f(iy)|^2; \qquad z = x + iy,$$

is given by $c \sin az$ where c is an arbitrary constant and a is constant, either real or purely imaginary. The special case $c = 1/a$, $a \to 0$ yields a particular solution, namely z itself. The proof consists in using the fact that $\log |f(x + iy)|$ is an harmonic function the Laplacian of which is zero. The resulting differential equations are integrable by quadratures. The method is easily extended to similar functional equations.

13. Dr. Einar Hille: *A class of functional equations.* Preliminary communication.

The problem here considered is to determine the analytic solutions of the functional equation

$$w = R(u, v); \quad w = |f(x+iy)|; \quad u = |f(x)|; \quad v = |f(iy)|,$$

where x and y are real variables and R is a rational function of u and v, satisfying certain conditions of symmetry and reality. Observing that the Laplacian of log R must vanish identically, we obtain a differential equation of the form $G_1u'' + G_2(u')^2 + G_3v'' + G_4(v')^2 = 0$ where G stands for a rational function of u and v with constant coefficients. Keeping y (or x) constant, u (or v) can be determined by three quadratures. If this process gives an analytic solution, $f(z)$, of the functional equation that is *single-valued* throughout the plane, then $f(z)$ is an *elliptic function* with one real and one pure imaginary period, or else a degenerate case of such a function, or, finally, a function of the form exp. $(az^2 + bz)$. The classification of the many-valued solutions is not yet completed. The method is easily extended to the case when R is an algebraic function of u and v.

14. Dr. Einar Hille: *Oscillation theorems in the complex domain.*

This paper is devoted to the study of the distribution in the complex plane of the zeros of solutions of a linear homogeneous differential equation of the second order, the coefficients of which are analytic functions of a complex variable. An integral equality, corresponding to Green's formula in the real theory, is deduced, by means of which it is possible to determine regions in the complex plane in which a given solution is different from zero as well as its first derivative. Various types of such regions are considered. The paper also contains a study of the asymptotic distribution of the zeros of a solution in the neighborhood of an irregular singular point of rather general type. The paper will appear in the TRANSACTIONS OF THIS SOCIETY.

15. Professor W. L. Crum: *Note on the internal evidence of the reliability of a test.*

The paper seeks to discover the theoretical limitations on the use of the common method of computing a reliability coefficient by comparison of the results for the odd- and evennumbered questions of the test. It is shown that this method is strictly applicable only to tests which are known to have very special degrees of uniformity, and that it is likely to lead to altogether inaccurate conclusions as to reliability for the irregular tests which we commonly give.

16. Professor W. L. Crum: *The use of the median in determining indices of seasonal variation.*

The paper suggests that the distribution of the monthly variables, in the usual historical series, about their respective averages has such a form that the arithmetic mean is a less reliable measure of the average tendency than is the median. A study is made of a particular historical economic series, known to possess marked and fairly regular seasonal variation, with a view to verifying the hypothesis. The conclusion appears unmistakable that the median should be used.

17. Professor R. M. Mathews: *A general construction for circular cubics.*

Let Q and R be two points on a variable circle which is cut again in S and T by two arbitrary fixed lines s and t through Q and R, respectively. The variable line ST cuts an arbitrary fixed line g in U. The variable line PU, drawn from a fixed point P, cuts the circle in A and B. The locus of these two points for the pencil of circles through Q and R is a circular cubic curve. Conversely, every circular cubic may be constructed in this manner.

18. Professor R. M. Mathews: *A theorem on conics, with applications.*

Three conics of the coaxial set $\{K\}$ on two points R and S cut an arbitrary fixed conic C in three sets of four points $\{A_i\}$, $\{B_i\}$, $\{C_i\}$, $(i = 1, 2, 3, 4)$, respectively; the conic of the set through $A_iB_iC_i$ cuts C in a fourth point D_i. Then the set of four points $\{D_i\}$ lies on a conic through R and S.

When R and S are the circular points at infinity the set $\{K\}$ consists of the circles of the plane; when the figure now is inverted we obtain theorems about groups of concyclic points on circular cubics and on bicircular quartics.

19. Mr. Louis Weisner: *A property of the characteristic elements of a group.*

If a group G can be generated by an invariant element S

and an invariant subgroup H, there exists an automorphism of G in which each element of H corresponds to itself and S corresponds to S^i, where i is a certain integer different from unity and prime to the order of S. The only exception to this statement is the case where the order of S is twice an odd number and H is of index 2 in G. It follows that a characteristic element of G whose order is not twice an odd number cannot appear in any possible set of independent generators of G, and is therefore contained in the ϕ-subgroup of G.

20. Dr. E. L. Post: *Visual intuition in Lobachevsky space.*

According to Klein, the development of non-euclidean geometry has proceeded through three stages, the synthetic stage centering around Lobachevsky, the differential geometry stage initiated by Riemann, and the projective measurement stage developed by Cayley and Klein. The present paper may be said to belong to a fourth stage, already found in the work of Poincaré. It takes an observer brought up in euclidean space, immerses him in Lobachevsky space, and relates what he sees there. In particular, the observer views the Lobachevsky straight line from all positions, and notes that in general it has the appearance of one branch of a hyperbola, always spreading away from him, and varying in size and shape with his distance from the line. He then observes objects at various distances from him, and notes that for a given distance an object appears much smaller in Lobachevsky than in euclidean space. Another interpretation is that Lobachevsky space is much roomier than euclidean space as one goes out from a given fixed center. These two developments of the Lobachevsky intuition are then related by showing that the shortest line between two points as determined by the metric presents the appearances described above.

21. Professor Elizabeth B. Cowley: *Note on a generalization of the old puzzle of* 8, 5, *and* 3 *pint vessels.*

Bachet (*Problèmes Plaisants & Delèctables*, 5th edition, Paris, 1884) has a discussion of a generalization of the old puzzle to obtain 4 pints of liquid from an 8 pint vessel full of liquid by the use of 2 empty vessels with capacities of 5 and 3 pints. Attention is called to the case in which $A < (B + C)$ (where A, B, C represent the capacities of the largest, middle, and smallest vessels respectively). Two examples are given: 20, 13, 9 and 16, 12, 7, and the statement is made that the first is possible and the second impossible but that these facts are not known *a priori*. In this paper, criteria are worked out for ascertaining *a priori* whether a solution is impossible when $A < (B + C)$. Incidentally, some properties of cases in which $A = B + C$ are noted.

R. G. D. RICHARDSON,
Secretary.

THE OCTOBER MEETING OF THE SAN FRANCISCO SECTION

The fortieth regular meeting of the San Francisco Section of the American Mathematical Society was held at the University of California, October 21, 1922. Professor Allardice presided during the early part of the meeting and Professor Cajori during the latter part. The total attendance was thirty-five, including the following twenty-four members of the Society:

Alderton, Allardice, Andrews, Barter, Bernstein, Blichfeldt, Buck, Cajori, Edwards, Growe, Haskell, Hoskins, Irwin, Lehmer, Levy, Libby, Moreno, F. R. Morris, T. M. Putnam, Shane, P. Sperry, Stromquist, A. R. Williams, Wong.

The following officers were chosen for the year: Chairman, Professor Florian Cajori; Secretary, Professor B. A. Bernstein; Programme Committee, Professors H. F. Blichfeldt, D. N. Lehmer, B. A. Bernstein.

It was decided to hold the next Fall meeting of the Section on Saturday, October 20, 1923, at the University of California.

Titles and abstracts of papers read at this meeting follow. The papers of Professors Bell and Smail were read by title.

1. Dr. J. D. Barter: *Spiral functions.* Preliminary report.

If α is a root of the equation $a_0 + a_1x + a_2x^2 + \cdots + \lambda + a_nx^n = 0$, termed the "fundamental equation," then exp. $(\alpha t) = f_0(t) + \alpha f_1(t) + \alpha^2 f_2(t) + \cdots + \lambda + \alpha^{n-1} f_{n-1}(t)$. The functions f are termed spiral functions. Their general properties and applications are summarized. Methods for calculating the values of these functions in special cases are outlined.

2. Professor E. T. Bell: *Applications of analysis to the arithmetic of higher forms.*

In papers published in the TRANSACTIONS OF THIS SOCIETY, it has been shown that any identity between elliptic, abelian or theta functions gives rise to a relation between functions wholly arbitrary except as to parity, the arguments of the functions being linear functions of the indeterminates representing an arbitrary integer in a *quadratic* form in several variables. In the present paper the restriction that the forms be of the *second* degree is removed. The method of paraphrase is thus widely generalized.

3. Professor E. T. Bell: *Umbral symmetric functions and algebraic analogues of the Bernoullian and Eulerian numbers and functions.*

The numbers of Bernoulli and Euler of rank n are replaced by rational integral algebraic functions in n independent variables. For unit values of all the variables the functions degenerate to the numbers of the same rank; the general relations between the numbers are special cases of relations between the functions. This paper is complementary to another, to be published in the TRANSACTIONS OF THIS SOCIETY, in which the numbers of rank n are replaced by polynomials of degree n in one variable.

4. Professor B. A. Bernstein: *An arithmetic representation of Boolean logic.*

The author shows how Boolean logic can be represented by arithmetic operations. The complete paper will appear in a later number of this BULLETIN.

5. Professor B. A. Bernstein: *Arithmetic independence systems for the Whitehead-Huntington postulates for Boolean algebras.*

The author exhibits simple arithmetic systems which may take the place of the abstract systems used by Huntington in proving the independence of his first set of postulates for Boolean algebras. The complete paper will appear in a later number of this BULLETIN.

6. Professor B. A. Bernstein: *A Boolean representation of a number field.*

In this paper the author gives a representation of a two-element field by means of operations of a Boolean algebra, and adds a note on the existence of abelian groups in any Boolean algebra. The complete paper will appear in a later number of this BULLETIN.

7. Professor Florian Cajori: *Origin of the symbols for "degrees, minutes, and seconds."*

The signs °, ′, ″ originated in the sixteenth century in the use of the exponential concept to operations involving sexagesimal numbers. The sign ° has been traced back to J. Peletier (1558).

8. Professor Florian Cajori: *The St. Andrew's Cross* × *as a mathematical symbol.*

The claim made by certain writers that × was adopted as the sign of multiplication on the European continent before the English adopted it in 1618 is found to be invalid. Before 1618 other uses (ten different ones) were made of two obliquely intersecting lines. This paper will appear in the MATHEMATICAL GAZETTE.

9. Professor Florian Cajori: *Origin of the names arithmetical and geometrical progression and proportion.*

Though not used by Euclid, Apollonius and Archimedes, these names originated earlier, in the time of Archytas and Hippasus; they met with more common adoption 400 years after Euclid. Aristotle uses the term "geometric proportion." The names indicate the fields of Greek mathematics in which the respective proportions found their widest application. This paper will be published in SCHOOL SCIENCE AND MATHEMATICS.

10. Professor L. L. Smail: *Applications of vector analysis to celestial mechanics.* Preliminary report.

The author has applied the methods of vector analysis to the problem of two bodies and to the discussion of the general integrals of the problem of n bodies, and has obtained some simplifications of treatment over the usual methods.

11. Dr. A. R. Williams: *Note on the form of a space curve in the vicinity of a singular point.* Preliminary report.

Von Staudt noted that the projections of a space curve on the three principal planes could be deduced from simple geometrical considerations. Christian Wiener prepared models illustrating the eight possible cases. The analytic side of the question has been treated at some length by several writers. The author shows that a simple analysis can be made by means of the torsion function.

12. Professor L. M. Hoskins: *Note concerning estimates of the rigidity of the earth.*

Estimates of the rigidity of the earth are based upon two independent lines of observational evidence, one showing the effect of tidal forces in changing the direction of apparent gravity, the other the prolongation of the earth's period of free nutation attributed to changes in centrifugal forces as the axis of rotation changes. Estimates based upon the two kinds of evidence have not been in close agreement. The author has shown (TRANSACTIONS OF THIS SOCIETY, vol. 21) that the two kinds of evidence may be harmonized by perfecting the theory of strain of an elastic sphere so as to take account of variation of density and elastic moduli. In the computations leading to this conclusion, however, a law of variation of density was for simplicity used which involved a considerable error in the value of the surface density as well as an error of about 3 per cent in the moment of inertia. Computations have now been made with a density formula which corrects these discrepancies and causes no important change in the quantities upon which estimates of rigidity are based. The formula used agrees rather closely with Laplace's law of density.

B. A. BERNSTEIN,
Secretary of the Section.

THE FRANK NELSON COLE PRIZE IN ALGEBRA

At the time of the retirement in December, 1920, of Professor F. N. Cole as Secretary of the American Mathematical Society and editor of its BULLETIN, a sum of money was collected from members of the Society by a committee, of which Professor H. S. White was chairman, and was presented to Professor Cole in recognition of his distinguished services through a period of twenty-five years. At the next meeting of the Council, Professor Cole donated this fund to the Society, to be used as the Council might think best. The committee, consisting of Professors Frank Morley (chairman), T. S. Fiske, and H. S. White, appointed at that meeting to make recommendations to the Council as to the use of the income of this fund, has presented a report recommending that this fund shall be used to endow a prize, to be called the Frank Nelson Cole Prize in Algebra. The recommendations of this committee, which were accepted by the Council at its meeting in October, 1922, follow below:

(1) The fund shall accumulate until by interest and contributions it reaches the amount of one thousand dollars.

(2) At the end of every five years thereafter the Council shall award from the available income not more than two hundred dollars as a prize for the best memoir offered in competition upon some question in the theory of Galois groups, or the theory of numbers, or some other part of algebra.

(3) The question or subject for competitive memoirs shall be announced by the Council at least two years in advance of the date for their submission; and the details of the mode of award shall be published at the same time.

(4) The prize shall be designated as the Frank Nelson Cole Prize in Algebra. At the time of each award the names of all previous recipients of the prize shall be published in the Bulletin of the Society, together with the name of the successful competitor and an abstract of his memoir.

(5) The prize may be withheld at the end of any quinquennium if no sufficiently meritorious memoir be submitted; and in such case a double prize may be awarded in the next following quinquennium.

(6) These conditions may be altered at any time by a two-thirds vote, in writing, of all members of the Council.

R. G. D. RICHARDSON, *Secretary.*

PERIODIC SOLUTIONS IN THE PROBLEM OF THREE BODIES*

BY F. H. MURRAY

1. *Introduction.* In *Les Méthodes Nouvelles de la Mécanique Céleste,* § 48, Poincaré gave a discussion of the periodic solutions of the third kind in the problem of three bodies, which has formed the basis for later researches by Poincaré and other writers. It is the purpose of this note to give an alternative demonstration of the existence of these solutions, using the methods of a preceding paragraph, and certain results of von Zeipel.

2. *Statement of Problem.* Suppose the number of degrees of freedom reduced to four by the methods of § 16,† and the variables of § 18 introduced. Then if

$$(1)\qquad \begin{cases} x_1 = \Lambda, & x_2 = \Lambda', & x_3 = \xi, & x_4 = \xi', \\ y_1 = \lambda, & y_2 = \lambda', & y_3 = \eta, & y_4 = \eta', \end{cases}$$

the equations of motion become,

$$(2)\qquad \frac{dx_i}{dt} = \frac{\partial F}{\partial y_i}, \qquad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i}, \qquad (i = 1, 2, 3, 4).$$

The function F satisfies the relations

$$(3)\qquad \begin{cases} F = F_0 + \mu F_1 + \cdots, \\ F_0 = F_0(x_1, x_2), \qquad \dfrac{\partial^2 F_0}{\partial x_1^2}\dfrac{\partial^2 F_0}{\partial x_2^2} - \left(\dfrac{\partial^2 F_0}{\partial x_1 \partial x_2}\right)^2 \neq 0, \end{cases}$$

in a domain $|x_1 - x_1^0| < b$, $|x_2 - x_2^0| < b$. Also,

$$F_1 = \sum A^{m_1 m_2}_{\alpha_1\alpha_2\beta_1\beta_2} x_3^{\alpha_1} x_4^{\alpha_2} y_3^{\beta_1} y_4^{\beta_2} \cos(m_1 y_1 + m_2 y_2 + h),$$

where $A^{m_1 m_2}_{\alpha_1\alpha_2\beta_1\beta_2}$ is analytic in x_1, x_2 in the domain considered.

We are now in a position to apply the method of § 46. In the notation of that paragraph, the coordinates of a periodic solution satisfy the conditions

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*Presented to the Society, September 8, 1922.

† Paragraph numbers refer to the paragraphs of Poincaré's *Les Méthodes Nouvelles,* vol. I.

$$\begin{aligned}
&\frac{\psi_i}{\mu} = \int_0^T \left(\frac{\partial F_1}{\partial y_i} + \mu\frac{\partial F_2}{\partial y_i} + \cdots\right) dt, && (i = 1, 2, 3, 4),\\
&-\frac{\partial F_0}{\partial x_i^0} T + \psi_i' = -\int_0^T \frac{\partial F}{\partial x_i}\, dt, && (i = 1, 2),\\
&\frac{\psi_i'}{\mu} = -\int_0^T \left(\frac{\partial F_1}{\partial x_i} + \mu\frac{\partial F_2}{\partial x_i} + \cdots\right) dt, && (i = 3, 4),
\end{aligned} \tag{4}$$

$$\frac{\psi_i}{\mu} = \psi_1' = \psi_2' = \frac{\psi_3'}{\mu} = \frac{\psi_4'}{\mu} = 0, \qquad (i = 1, 2, 3, 4). \tag{5}$$

These equations are to be solved for the values of the coordinates (x_i, y_i), at the time $t = 0$, and for $|\mu| < a$, where a is arbitrarily small.

3. *Solution of the Equations.* On account of the integral $F = \text{const.}$, one of the equations, for instance $\psi_2 = 0$, may be omitted, and the constant $\bar{\omega}_2$ chosen identically zero. If $\mu = 0$, equations (5) become

$$\frac{\partial R}{\partial \bar{\omega}_1} = \frac{\partial R}{\partial \bar{\omega}_3} = \frac{\partial R}{\partial \bar{\omega}_4} = \psi_1' = \psi_2' = \frac{\partial R}{\partial x_3^0} = \frac{\partial R}{\partial x_4^0} = 0, \tag{5'}$$

$$R = \frac{1}{T}\int_0^T F_1 dt \cdot$$

The Jacobian of this system of functions is equal to the product of the Hessian of F_0 with respect to x_1^0, x_2^0 by the Hessian of R with respect to $\bar{\omega}_1$, $\bar{\omega}_3$, $\bar{\omega}_4$, x_3^0, x_4^0, multiplied by a non-vanishing factor; from (3) it only remains to show that the latter, or the Hessian of R with respect to λ^0, ξ, ξ', η, η', is different from zero if x_1^0, x_2^0, λ^0 are suitably chosen, and $\xi = \xi' = \eta = \eta' = 0$.

This Hessian is equal, except for a non-vanishing factor, to the determinant D_2 of von Zeipel* (with $\theta = 0$). It can easily be verified that for certain systems of values of x_1^0, x_2^0, λ^0, $D_2 \neq 0$; the existence of periodic orbits of the corresponding types follows.

THE UNIVERSITY OF WEST VIRGINIA

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* *Recherches sur les solutions périodiques de la troisième sorte dans le problème des trois corps,* NOVA ACTA R. SOCIET. UPS., (3), vol. 20, p. 51.

NOTE ON QUARTILES AND ALLIED MEASURES*

BY DUNHAM JACKSON

If a number of values $a_1, a_2, \cdots, a_n$ of a quantity x have been observed, the lower quartile of this set of observations may be roughly described as a number x_1 such that one fourth of the a's are less than x_1 and three fourths of them are greater than x_1. Something more is needed for an exact definition, inasmuch as the condition stated either leads to an indeterminate value or is impossible of realization, according to circumstances. If x_1 is defined as a value of x which reduces to a minimum the expression

$$S_1 = \sum_{i=1}^{n} \varphi_1(x - a_i),$$

where $\varphi_1(x) = \frac{3}{4}x$ for $x \geqq 0$, $\varphi_1(x) = -\frac{1}{4}x$ for $x \leqq 0$, there will always be at least one value of x_1 satisfying the condition, and this will agree with the value of the quartile as ordinarily understood, but if $n = 4k$ and $a_k \neq a_{k+1}$, when the a's are arranged in order of increasing algebraic magnitude, any number between a_k and a_{k+1} will meet the requirement. It is the purpose of this note to show that a unique determination results in all cases from a definition analogous to one which the author recently suggested for the median.† As in the previous instance, the definition is admittedly of theoretical rather than practical interest. The discussion is put in such a form as to apply equally well to an arbitrary percentile or other measure of similar character, the ratio 1 : 3 being replaced by any other positive ratio.

Let $a_1, \cdots, a_n$ be a set of real numbers (not necessarily all distinct) arranged in ascending order of magnitude algebraically, and let c be an arbitrary number of the interval $0 < c < 1$. For $p \geqq 1$, let a function $\varphi_p(x)$ be defined as follows:

$$\varphi_p(x) = (1 - c)x^p \text{ for } x \geqq 0, \qquad \varphi_p(x) = c(-x)^p \text{ for } x \leqq 0.$$

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* Presented to the Society, October 28, 1922.

† *Note on the median of a set of numbers*, this BULLETIN, vol. 27 (1920–21), pp. 160–164.

The value of c will be kept constant throughout, and need not be indicated in the notation for the function φ. Let

$$S_p = \sum_{i=1}^{n} \varphi_p(x - a_i),$$

and let x_p stand for the value of x, or any value of x, which reduces S_p to a minimum. It is to be shown that x_p is uniquely determined for each value of $p > 1$; that as p approaches 1, x_p approaches x_1, if x_1 has a determinate value; and that in the contrary case, x_p approaches a definite limit belonging to the interval within which x_1 is indeterminate.

Consider first the case that cn is not an integer. Let k be the integer such that $k - 1 < cn < k$. Then x_1 definitely has the value a_k. For as x changes from a_k to $a_k + \delta$, at least k terms of the sum S_1 are increased, each by the amount $(1 - c)\delta$, and not more than $n - k$ terms are diminished, the amount of decrease in each case being $c\delta$ at most, so that the net change in S_1 is at least

$$k(1 - c)\delta - (n - k)c\delta = (k - cn)\delta > 0;$$

and as x changes from a_k to $a_k - \delta$, at least $n - k + 1$ terms are increased, not more than $k - 1$ are diminished, and the net change is at least

$$(n - k + 1)c\delta - (k - 1)(1 - c)\delta = (cn - k + 1)\delta > 0.$$

It will be shown that x_p is uniquely determined for each value of $p > 1$, and that $\lim_{p=1} x_p = a_k$.

When $p > 1$, the function $\varphi_p(x)$ is continuous and has a continuous derivative for all values of x, including $x = 0$. Since $S_p(x)$ is continuous and becomes infinite as x becomes infinite in either direction, it must have at least one minimum. A necessary condition for a minimum is the vanishing of $S_p'(x)$. But it is readily seen, either by inspection or by writing down the explicit formulas, that $\varphi_p'(x)$ always increases when x increases, so that $S_p'(x)$ likewise increases when x increases, and can vanish only once. This proves the existence and uniqueness of x_p.

Let ϵ be an arbitrarily small positive quantity, and let r be an index such that $a_r < a_k + \epsilon \leqq a_{r+1}$; it is clear that $r \geqq k$.

It follows from the definition of φ that $\varphi_p'(x) = (1-c)px^{p-1}$ or $-cp(-x)^{p-1}$, according to the sign of x, and hence

$$\frac{1}{p}S_p'(a_k+\epsilon) = (1-c)(a_k+\epsilon-a_1)^{p-1}+\cdots$$
$$+(1-c)(a_k+\epsilon-a_r)^{p-1} - c(a_{r+1}-a_k-\epsilon)^{p-1} - \cdots$$
$$- c(a_n-a_k-\epsilon)^{p-1}.$$

When p approaches 1, each of the first r terms on the right, apart from the factor $1-c$, approaches the limit 1, and each of the remaining terms, apart from the factor c, but inclusive of the algebraic sign, approaches -1 or possibly 0. So

$$\lim_{p=1} S_p'(a_k+\epsilon) \geqq r(1-c)-(n-r)c = r-cn \geqq k-cn > 0.$$

Similarly,

$$\lim_{p=1} S_p'(a_k-\epsilon) < 0.$$

For if the definition of r is changed so that $a_r \leqq a_k - \epsilon < a_{r+1}$, then $r \leqq k-1$, and

$$\frac{1}{p}S_p'(a_k-\epsilon) = (1-c)(a_k-\epsilon-a_1)^{p-1}+\cdots$$
$$+(1-c)(a_k-\epsilon-a_r)^{p-1} - c(a_{r+1}-a_k+\epsilon)^{p-1} - \cdots$$
$$- c(a_n-a_k+\epsilon)^{p-1},$$

$$\lim_{p=1} S_p'(a_k-\epsilon) \leqq r(1-c)-(n-r)c = r-cn$$
$$\leqq k-1-cn < 0.$$

So the value x_p for which S_p' vanishes must be between $a_k-\epsilon$ and $a_k+\epsilon$ when p is sufficiently near 1, or, in other words, $\lim_{p=1} x_p = a_k$.

Suppose now that cn is an integer, and let $cn = k$. If it happens that $a_k = a_{k+1}$, reasoning similar to that presented above shows that $x_1 = a_k$ and $\lim_{p=1} x_p = a_k$, as before.

This special case being left aside, it is to be assumed that $a_k < a_{k+1}$. The definition of x_1 is satisfied by a_k or a_{k+1} or any intermediate value. For $p > 1$, on the other hand, x_p is seen to be uniquely determined, by the same argument as was used before. Furthermore, it is recognized that

$$\lim_{p=1} S_p'(a_k) = (k-1)(1-c)-(n-k)c$$
$$= k-1+c-cn = c-1 < 0,$$

$$\lim_{p=1} S_p'(a_{k+1}) = k(1-c)-(n-k-1)c = k-cn+c = c > 0,$$

so that $a_k < x_p < a_{k+1}$ when p is sufficiently near 1. It remains to be shown that x_p approaches a definite limit as p approaches 1.

Let x have a value between a_k and a_{k+1}. For $i = 1, 2, \cdots, k$, $x - a_i$ is positive, and

$$\begin{aligned}(x - a_i)^{p-1} &= e^{(p-1)\log(x-a_i)}\\ &= 1 + (p-1)\log(x-a_i) + \tfrac{1}{2}(p-1)^2 \log^2(x-a_i) + \cdots\\ &= 1 + (p-1)\log(x-a_i) + (p-1)^2\rho_i(x,p),\end{aligned}$$

where $\rho_i(x, p)$ is a function which approaches $\frac{1}{2}\log^2(x - a_i)$, and so remains finite, if x is held fast and p approaches 1. For $i > k$,

$$(a_i - x)^{p-1} = 1 + (p-1)\log(a_i - x) + (p-1)^2\rho_i(x,p),$$

where $\rho_i(x, p)$ again remains finite for fixed x as p approaches 1. If these values are substituted in the explicit expression for $(1/p)S_p'(x)$, there will be k terms each equal to $(1 - c)$ and $(n - k)$ terms each equal to $(- c)$, which will destroy each other, because of the relation $cn = k$, and each of the remaining terms will have a factor $p - 1$, so that we may write

$$\begin{aligned}\frac{1}{p(p-1)} S_p'(x) &= (1-c)[\log(x-a_1) + \cdots + \log(x-a_k)]\\ &\quad - c[\log(a_{k+1}-x) + \cdots + \log(a_n - x)] + (p-1)\rho(x,p)\\ &= \log\frac{[(x-a_1)\cdots(x-a_k)]^{1-c}}{[(a_{k+1}-x)\cdots(a_n-x)]^c} + (p-1)\rho(x,p),\end{aligned}$$

the function ρ remaining finite as p approaches 1.

As the exponents c and $1 - c$ are both positive, the fraction on the right increases steadily from 0 to $+\infty$ as x goes from a_k to a_{k+1}, and the logarithm increases steadily from $-\infty$ to $+\infty$, taking on the value 0 just once, say for $x = X$. For $x = X + \epsilon$, the logarithm is positive and independent of p, while the term $(p - 1)\rho(X + \epsilon, p)$ approaches zero as p approaches 1. So the value of the whole expression on the right is positive when $p - 1$ is sufficiently small. For a similar reason, the expression is negative for $x = X - \epsilon$, if p is sufficiently close to 1. This means that the root of $S_p'(x)$ is between $X - \epsilon$ and $X + \epsilon$ when $p - 1$ is sufficiently small, that is,

$$\lim_{p=1} x_p = X.$$

THE UNIVERSITY OF MINNESOTA

RULED SURFACES WITH DIRECTOR PLANES*

BY J. K. WHITTEMORE

It is well known that one family of asymptotic lines of a ruled surface is composed of the rulings or generating straight lines of the surface; that the four points in which a variable ruling is cut by any four members of the other family of asymptotic lines—hereinafter called the curved asymptotic lines—are in constant cross ratio.† In this paper we prove, using this theorem of the constant cross ratio, that the necessary and sufficient condition that a ruled surface have the property that the segments cut from a variable ruling by any three curved asymptotic lines are in constant ratio is that the surface have a director plane; that the necessary and sufficient condition that a ruled surface with a director plane have the property that any two curved asymptotic lines cut a constant length from a variable ruling is that the surface have a constant parameter of distribution. The latter type of ruled surface is further discussed: it is shown that these surfaces bear a close analogy to the right helicoid; that given any right helicoid the analogous surfaces are obtained by the choice of an arbitrary function of one variable and by two quadratures; some properties of the curved asymptotic lines are given; finally a property of the line of striction of a ruled surface with a director plane is proved—a property which is not however characteristic of these surfaces.

Suppose the rectangular coordinates of a point of any ruled surface are given by the equations

$$\xi = x + ul, \qquad \eta = y + um, \qquad \zeta = z + un, \tag{1}$$

where x, y, z are the coordinates of a point of the directrix and are regarded as functions of a parameter v; l, m, n, the direction cosines of the ruling through the point v of the directrix, are also functions of v and u is length along the ruling measured from the directrix. The equation of any curve of the surface other than a ruling is given by writing in (1) u equal to a function of v. Suppose u_1, u_2, u_3, u_4 are four different functions of v giving the equations of any four curved asymptotic lines; then the cross ratio

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* Presented to the Society, April 23, 1921.

† Eisenhart, *Differential Geometry*, p. 249. The case of developable surfaces, where the two families of asymptotic lines coincide, is excluded in this statement and throughout the following discussion.

$$\frac{u_1 - u_3}{u_2 - u_3} \div \frac{u_1 - u_4}{u_2 - u_4},$$

is constant. If u_1, u_2, u_3 give any three curved asymptotic lines, the equation of any such line is

$$\frac{u_1 - u_3}{u_2 - u_3} \div \frac{u_1 - u}{u_2 - u} = c, \tag{2}$$

where c is an arbitrary constant. It is evident that the necessary and sufficient condition that the ratio $(u_1 - u_3)/(u_2 - u_3)$ be constant for any three curved asymptotic lines is that (2) be satisfied by writing u equal to infinity, in other words that the necessary and sufficient condition that any three curved asymptotic lines cut from a variable ruling segments in a constant ratio is that the intersection of the surface with the plane at infinity be an asymptotic line. We proceed to show that this condition is equivalent to the requirement that the surface have a director plane. The differential equation of the asymptotic lines of any surface is

$$Ddu^2 + 2D'dudv + D''dv^2 = 0. \tag{3}$$

From (1) we find $D = 0$, so verifying the statement that the rulings, v constant, form one family of asymptotic lines on any ruled surface. We find further

$$\begin{aligned} HD' &= \begin{vmatrix} l' & m' & n' \\ l & m & n \\ x' & y' & z' \end{vmatrix}, \\ HD'' &= \begin{vmatrix} x''+ul'' & y''+um'' & z''+un'' \\ l & m & n \\ x'+ul' & y'+um' & z'+un' \end{vmatrix} = Lu^2+Mu+N. \end{aligned} \tag{4}$$

To find the condition for an asymptotic line in the plane at infinity we set $u = p/q$ and rewrite equation (3)

$$2D'(qdp - pdq) + (Lp^2 + Mpq + Nq^2)dv = 0.$$

In order that $q = 0$ satisfy this equation it is necessary and sufficient that

$$L = \begin{vmatrix} l'' & m'' & n'' \\ l & m & n \\ l' & m' & n' \end{vmatrix} = 0.$$

That L vanish for all v is the condition that there exist a relation with constant coefficients, A, B, C, not all zero,

$$Al + Bm + Cn = 0.$$

It follows that the rulings are perpendicular to a fixed direction, that is, that the surface has a director plane. It is moreover evident that the asymptotic line at infinity is the intersection of the director plane with the plane at infinity and is therefore a straight line. It is easy to verify the property proved for any ruled surface with a director plane, at the same time finding by a single quadrature, in conformity to the general theory of the Riccati equation, the finite equation of all curved asymptotic lines when one such line is known in addition to the line at infinity. We may without restriction choose the yz plane as the director plane and also choose the parameter v as x, since we need not consider the plane surface x constant. We then write

$$l = 0, \qquad m = \sin\varphi, \qquad n = \cos\varphi,$$

and equations (1) and (4) may be written

(1′) $\xi = x, \qquad \eta = y + u\sin\varphi, \qquad \zeta = z + u\cos\varphi;$

(4′) $HD' = \varphi', \qquad HD'' = u\varphi'' + y''\cos\varphi - z''\sin\varphi.$

For the curved asymptotic lines (3) becomes

$$2\varphi' du + (u\varphi'' + y''\cos\varphi - z''\sin\varphi)dv = 0.$$

If u_1, u_2, u_3 are three solutions of this equation we find

(5) $(u_1 - u_2)^2 = c_2/\varphi', \qquad (u_1 - u_3)^2 = c_3/\varphi',$

so verifying the property of proportional segments. If u_1 is known, all curved asymptotic lines are given by $(u_1 - u)^2 = c/\varphi'$ where c is an arbitrary constant. The parameter of distribution of the surface (1′) is found to be $1/\varphi'$.*

Consider now a special case of ruled surfaces with director planes, surfaces such that any two curved asymptotic lines cut a constant length from a variable ruling. It is evident from equations (5) that it is necessary and sufficient in order that the surface (1′) have the required property that φ' be constant, that is, that the parameter of distribution be constant. The case $\varphi' = 0$ may be excluded since for this value the surface (1′) is a cylinder. We may write $\varphi = x/a + b$ and by a proper choice of origin take $b = 0$. The equations of the required surfaces are

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* Eisenhart, loc. cit., p. 246.

$$(1'') \qquad \xi = x, \qquad \eta = y + u \sin\frac{x}{a}, \qquad \zeta = z + u \cos\frac{x}{a} \cdot$$

The ruling x of (1″) is parallel to the ruling x of the right helicoid

$$\xi = x, \qquad \eta = u \sin\frac{x}{a}, \qquad \zeta = u \cos\frac{x}{a} \cdot$$

We remark that the surface (1″) is not developable for the total curvature is $-1/(a^2H^4)$. If we suppose that the directrix, $u = 0$, of (1″) is an asymptotic line we may prove certain properties of these surfaces, in particular of the curved asymptotic lines, u constant. This supposition imposes the condition on the functions y and z

$$(6) \qquad y'' \cos\frac{x}{a} - z'' \sin\frac{x}{a} = 0.$$

When a is given, that is, when the analogous right helicoid is given, equation (6) is satisfied by choosing y as any function of x and determining z by two quadratures. If z_1 is one function so determined, every value of z found from (6) has the form $z = z_1 + bx + c$. In geometrical terms we may say that, given the analogous right helicoid, the projection of an asymptotic line on the xy plane may be chosen arbitrarily; the projection of this asymptotic line on the xz plane is then determined except for the addition of the ordinates of an arbitrary straight line. When the two projections of one curved asymptotic line of the surface are known the projections of any other curved asymptotic line are obtained by adding to the ordinates of the projection of the former the ordinates of two sine curves of equal amplitude, each of the period of the analogous right helicoid $2\pi a$, differing in phase in the two projections by a quarter period. If the surface contains a finite straight line other than the rulings, such a line is a member of the family of curved asymptotic lines, and the equations of such a surface are

$$\xi = x, \qquad \eta = Ax + B + u \sin\frac{x}{a},$$

$$\zeta = Cx + D + u \cos\frac{x}{a} \cdot$$

The projections of all curved asymptotic lines of this surface are curves of the "sinoid" type obtained by adding the ordinates of a sine curve to those of a straight line, familiar to students of Smith and Gale, *New analytic geometry*.*

The discovery of a surface of this kind in M. C. Foster's thesis, *Rectilinear congruences related to special surfaces*, as one of the principal surfaces of a congruence having a plane as a limit surface, led to the discussion contained in this paper.

The line of striction of a ruled surface with a director plane (1′) is† $u\varphi' = z' \sin \varphi - y' \cos \varphi$. If the line of striction is chosen as the directrix, $u = 0$, we obtain the theorem: A necessary and sufficient condition that a curve on a ruled surface with a director plane be the line of striction is that the projections on the director plane of the ruling and the tangent to the curve coincide for every point of the curve. It may be proved, conversely, that if the projections on any plane of the tangent to the line of striction of a ruled surface and of the ruling through that point of the line of striction coincide for all points, then, if the surface is not developable, all rulings make the same angle with the plane of projection. It may be easily proved from the preceding theorem that for no other ruled surface with a director plane than the right helicoid is the line of striction an asymptotic line.

YALE UNIVERSITY

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* Pp. 111, 112.

† Eisenhart, loc. cit., p. 243.

ON TRANSFORMABLE SYSTEMS AND COVARIANTS OF ALGEBRAIC FORMS*

BY C. C. MACDUFFEE

1. *Introduction.*—The purpose of this article is to give a rigorous demonstration of an important theorem in the theory of covariants of algebraic forms in p variables; namely, that if $(G_1, \ldots, G_h) = (0, \ldots, 0)$ is an invariantive property, the G_i being polynomials in the coefficients of the forms, there exists a set $V_1, \ldots, V_\nu$ of relative covariants in $p-1$ sets of cogredient variables, such that $(V_1, \ldots, V_\nu) = (0, \ldots, 0)$ when and only when $(G_1, \cdots, G_h) = (0, \cdots, 0)$. The corresponding theorem for invariants, i.e., for $h = 1$, is given by Bôcher, *Introduction to Higher Algebra* (p. 232). Bôcher there states that "a projective relation expressed by the identical vanishing of a covariant or contravariant is typical of what we shall usually have when a single equation is not sufficient to express the condition." This paper shows that such a projective relation can in general be characterized by the simultaneous vanishing of a number of covariants.

The special case of this theorem for binary forms is mentioned without proof by Clebsch, *Binäre Formen* (p. 91). J. P. Gram in the MATHEMATISCHE ANNALEN (vol. 7), and J. Deruyts in a book entitled *Essai d'une Théorie Générale des Formes Algébriques* (Brussels, 1891) consider the characterization by covariants of particular forms defined by the holding of identical relations among their coefficients. Both proofs are incomplete, however, and Gram's method actually leads to a false result in case the given conditions are non-homogeneous.

2. *Transformable Systems.* Consider a system of l algebraic forms in p variables

$$(1)\qquad f_i(a_{i1}, a_{i2}, \cdots a_{iq_i}; x_1, x_2, \cdots, x_p) \equiv f_i, \quad (i = 1, \cdots, l)$$

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* Presented to the Society, December 28, 1922.

of degree r_i in the variables. The linear homogeneous transformation

$$x_i = \sum_{j=1}^{j=p} \alpha_{ij} x_j', \qquad (i = 1, \cdots, p) \tag{2}$$

of determinant $|\alpha| \neq 0$ induces upon the coefficients of the forms (1) a system of transformations

$$a_{ij}' = \sum_{k=1}^{k=q_i} \beta_{ijk} a_{ik}, \qquad (j = 1, \cdots, q_i) \tag{3}$$

of determinants $|\beta_i| = |\alpha|^{k_i}$ where* k_i is an integer.

According to the usage of J. Deruyts,† a *transformable system* is defined as a system of linearly independent polynomials $F_1, \cdots, F_s$ in the coefficients a_{ij} which are transformed by (3) according to the law

$$F_i' = \sum_{j=1}^{j=s} \gamma_{ij} F_j, \qquad (i = 1, \cdots, s), \tag{4}$$

where the F_i' are the same functions of the primed coefficients a_{ij}' as the F_i are of the a_{ij}, and where the γ_{ij} are polynomials in the β_{ijk} and hence in the α_{ij}. In particular, (3) is a transformable system for every i.

It follows that the determinant $|\gamma|$ of (4) is a power of $|\alpha|$.‡ For if possible choose a set of numbers α_{ij} such that $|\alpha| \neq 0$ and $|\gamma| = 0$. It would then follow that the F_i' were linearly dependent, contrary to definition. Hence $|\gamma|$ is different from zero for all values of the α_{ij} for which $|\alpha| \neq 0$, and hence $|\gamma| = c|\alpha|^n$ where n is a positive integer or zero. Now choose $\alpha_{ij} = \delta_{ij}$ and (2) becomes the identity transformation. Then (3) and (4) likewise become identity transformations, so $c = 1$. The following theorem may now be proved.

THEOREM. *Let $G_1, \cdots, G_h$ be a system of polynomials in the a_{ij}, and $G_1', \cdots, G_h'$ the same functions of the a_{ij}' having the properties that*

(*a*) *It is possible to effect a transformation* (2) *making* $(G_1', \cdots, G_h') = (0, \cdots, 0)$ *when and only when* $(G_1, \cdots, G_h) = (0, \cdots, 0)$,

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* Hurwitz, *Zur Invarianttheorie*, MATHEMATISCHE ANNALEN, vol. 45, pp. 381–404.

† BULLETINS DE L'ACADÉMIE DES SCIENCES DE BELGIQUE, (3), vol. 32 (1896), p. 82. Deruyts requires that the F_i be homogeneous, and does not require that they be linearly independent.

‡ Proved more at length by Deruyts, loc. cit., p. 434

(b) If $(G_1, \cdots, G_h) = (0, \cdots, 0)$, *then* $(G_1', \cdots, G_h') = (0, \cdots, 0)$ *is true for all transformations* (2).

Then there exists a transformable system $(F_1, \cdots, F_s)$ *such that* $(F_1, \cdots, F_s) = (0, \cdots, 0)$ *when and only when* $(G_1, \cdots, G_h) = (0, \cdots, 0)$.

The case where $(G_1, \cdots, G_h) = (0, \cdots, 0)$ for no set of values of the a_{ij} may be disposed of first. In this case the transformable system may consist of the integer 1, which is never zero and is undisturbed by the transformation (2).

It is then permissible to assume that in the remaining case there actually exists at least one set of coefficients a_{ij} for which $(G_1, \cdots, G_h) = (0, \cdots, 0)$. If there are linear relations between the polynomials $G_1, \cdots, G_h$, we shall henceforth consider a subset $G_1, \cdots, G_g$ which are linearly independent and have the property that all the G_i which have been discarded are linear combinations with constant coefficients of them. Evidently $(G_1, \cdots, G_g) = (0, \cdots, 0)$ when and only when $(G_1, \cdots, G_h) = (0, \cdots, 0)$.

From relations (3) we can express $G_1', \cdots, G_g'$ in the form

$$G_i' = \sum_{j=1}^{j=t} \zeta_{ij} H_j, \qquad (i = 1, \cdots, g), \tag{5}$$

where the ζ_{ij} are polynomials in the β_{ijk} and therefore in the α_{ij}. If m represent the maximum degree of the G_i' in the a_{ij}', then the degree of every H_j in the a_{ij} will be $\leqq m$.

Choose the α_{ij} so that (2) becomes the identity transformation. Then (3) likewise become identity transformations and (5) becomes

$$G_i' = G_i = \sum_{j=1}^{j=t} c_{ij} H_j, \qquad (i = 1, \cdots, g),$$

where the c_{ij} are constants. Since the G_i are by hypothesis linearly independent, the rank of the matrix $(c_{ij}) = c$ is g. We assume that our notation is so chosen that the first g columns of the matrix form a determinant which is different from zero, and solve for $H_1, \cdots, H_g$ in terms of $G_1, \cdots, G_g$, $H_{g+1}, \cdots, H_t$. The relations (5) may, by the substitution of these values, be made to assume the form

$$G_i' = \sum_{j=1}^{j=g} \xi_{ij} G_j + \sum_{j=g+1}^{j=t} \eta_{ij} H_j, \qquad (i = 1, \cdots, g).$$

Moreover we may assume that there exists no linear relation with constant coefficients between the polynomials G_j and H_j, or between the columns of the matrix η; for in either case a condensation would be possible reducing the number of functions H_j.

It will now be shown that every H_j vanishes for such values of the a_{ij} as make $(G_1, \cdots, G_g)=(0, \cdots, 0)$. For if c_{ij} is any particular set of values of the a_{ij} which reduce $(G_1, \cdots, G_g)$ to $(0, \cdots, 0)$ and the H_j to constants $\overline{H}_j$, we have by hypothesis $(G_1', \cdots, G_g') = (0, \cdots, 0)$, and therefore

$$\sum_{j=g+1}^{j=t} \eta_{ij}\overline{H}_j = 0, \qquad (i = 1, \cdots, g).$$

Since the columns of this matrix are linearly independent, we have $\overline{H}_j = 0$ for $j = g + 1, \cdots, t$.

Since the set of polynomials $(G_1, \cdots, G_g, H_{g+1}, \cdots, H_t) = (0, \cdots, 0)$ when and only when $(G_1, \cdots, G_g)=(0, \cdots, 0)$, we denote the augmented set by $(G_1, \cdots, G_t)$ and obtain, just as before, the relation

$$G_i' = \sum_{j=1}^{j=t} \xi_{ij}'G_j + \sum_{j=t+1}^{j=u} \eta_{ij}'H_j, \quad (i=1, \cdots, t). \tag{6}$$

We consider this system of equations to be so reduced that no H_j is expressible linearly with constant coefficients in terms of the G_j and the remaining H_j, and that the columns of the matrix η' are linearly independent. Moreover, the degree of every H_j is $\leqq m$, the maximum degree of the original polynomials $G_1, \cdots, G_g$. Proceeding in this way, after a finite number of steps we reach a relation of the form (6) in which every H_j is identically zero; for there are but a finite number of linearly independent polynomials of degree $\leqq m$ in a finite number of variables a_{ij}. This final set of G's form our transformable system $F_1, \cdots, F_s$. For $(F_1, \cdots, F_s) = (0, \cdots, 0)$ when and only when $(G_1, \cdots, G_g) = (0, \cdots, 0)$, which is true when and only when $(G_1', \cdots, G_g')=(0, \cdots, 0)$, which in turn is true when and only when $(F_1', \cdots, F_s') = (0, \cdots, 0)$.

3. *Absolute Covariants.* The elements γ_{ij} of the matrix γ of the transformation (4) are polynomials in the elements α_{ij} of the transformation (2). Hence γ may be considered as a function $T(\alpha)$ of the matrix α. As was noted by Deruyts,*

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* Loc. cit., p. 437.

the matrix $\gamma = T(\alpha)$ of a transformable system of linearly independent polynomials obeys the functional equation

$$T(\alpha\beta) = T(\beta) \cdot T(\alpha).$$

Such matrices have been called *invariant matrices,* and the operation T an *invariant operation* by Schur,* who derived many of their properties.

If we denote by F the matrix

$$\begin{bmatrix} F_1 \\ \cdot \\ \cdot \\ \cdot \\ F_s \end{bmatrix},$$

then (4) may be written $F' = \gamma F$. If $p = (p_{ij})$ denote a non-singular constant matrix, and if we set $\overline{F} = pF$, then the functions $\overline{F}_i$ are linear combinations of the F_j having the property that $(\overline{F}_1, \cdots, \overline{F}_s) = (0, \cdots, 0)$ when and only when $(F_1, \cdots, F_s) = (0, \cdots, 0)$. From composition of transformations we have $\overline{F}' = pF' = p\gamma F = p\gamma p^{-1}\overline{F}$. Hence the matrix $p\gamma p^{-1}$ is *equivalent* to the matrix γ in the sense that there exists a set of functions $\overline{F}_i$ which are transformed by $p\gamma p^{-1}$ and which all vanish when and only when all the F_i vanish. Schur calls a matrix $T(\alpha)$ reducible if there exists a matrix equivalent to it which reduces into distinct blocks, i.e., if

$$pT(\alpha)p^{-1} = \begin{pmatrix} M_1 & \\ & M_2 \end{pmatrix}.$$

The blocks M_1 and M_2 are themselves invariant matrices.† An invariant matrix is reducible into irreducible invariant matrices in essentially but one way,‡ and the elements of an irreducible matrix are linearly independent homogeneous polynomials in the coefficients α_{ij} of the original transformation

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* *Ueber eine Klasse von Matrizen die sich einer gegebenen Matrix zuordnen lassen,* Berlin thesis, 1901, p. 5. In this paper the functional equation is taken as $T(xy) = T(x) \cdot T(y)$. The transposes of these matrices satisfy the equation given above.

† Loc. cit., p. 6.

‡ Loc. cit., p. 39.

(2).* An irreducible matrix cannot transform a set of linearly dependent functions except they all be zero.†

We now consider the matrix γ to be replaced by an equivalent matrix consisting of irreducible blocks. Suppose $\varphi = T(\alpha)$ is one of these blocks and K_i for $i = 1, \cdots, t$, linear combinations of the F_i which are transformed by this invariant matrix. By showing that the K_i are coefficients of a covariant V_1, we show that the condition $(F_1, \cdots, F_s) = (0, \cdots, 0)$ can be replaced by the equivalent condition $(V_1, \cdots, V_v) = (0, \cdots, 0)$.

It is evident from (2) that if $x_i^{(j)}$ are p systems of variables cogredient with the x_i, then

$$(7)\quad \begin{bmatrix} x_1^{(1)} & x_1^{(2)} & \cdots & x_1^{(p)} \\ x_2^{(1)} & x_2^{(2)} & \cdots & x_2^{(p)} \\ \cdot & \cdot & \cdot & \cdot \\ x_p^{(1)} & x_p^{(2)} & \cdots & x_p^{(p)} \end{bmatrix} = \begin{bmatrix} \alpha_{11} & \alpha_{12} & \cdots & \alpha_{1p} \\ \alpha_{21} & \alpha_{22} & \cdots & \alpha_{2p} \\ \cdot & \cdot & \cdot & \cdot \\ \alpha_{p1} & \alpha_{p2} & \cdots & \alpha_{pp} \end{bmatrix} \begin{bmatrix} x_1^{(1)\prime} & x_1^{(2)\prime} & \cdots & x_1^{(p)\prime} \\ x_2^{(1)\prime} & x_2^{(2)\prime} & \cdots & x_2^{(p)\prime} \\ \cdot & \cdot & \cdot & \cdot \\ x_p^{(1)\prime} & x_p^{(2)\prime} & \cdots & x_p^{(p)\prime} \end{bmatrix}.$$

Denoting these matrices by X and X' respectively, we have $X = \alpha X'$. Now since T is an invariant operation, where $T(\alpha) \doteq \varphi$, we have $T(X) = T(X') \cdot \varphi$, or, denoting the elements of $T(X)$, $T(X')$ by X_{ij}, X_{ij}', respectively, we have ‡

$$X_{ij} = \sum_{k=1}^{k=t} \varphi_{kj} X_{ik}', \qquad (i, j = 1, \cdots, t).$$

Since φ is irreducible, the functions X_{ij} are all linearly independent, for X_{ij} is the result of substituting $x_i^{(j)}$ for α_{ij} in the element of φ lying in the ith row and jth column. Now

$$\begin{aligned} \sum_{j=1}^{j=t} K_j X_{ij} &= \sum_{j=1}^{j=t} K_j \sum_{k=1}^{k=t} \varphi_{kj} X_{ik}' \\ &= \sum_{k=1}^{k=t} X_{ik}' \sum_{j=1}^{j=t} \varphi_{kj} K_j = \sum_{k=1}^{k=t} K_k' X_{ik}'. \end{aligned}$$

Therefore $\sum_{j=1}^{j=t} K_j X_{ij}$ is an absolute covariant of the system of forms (1) for every value of $i = 1, \cdots, t$.

4. *Relative Covariants.* We have shown that $\sum_{j=1}^{j=t} K_j X_{ij} \doteq V_i$ for $i = 1, \cdots, t$ is transformed into itself by every transformation of type (2). Moreover the X_{ij} are linearly independent functions of p sets of variables $x_i^{(j)}$ cogredient with x_i. We shall now show that *there exists a relative covariant with the same coefficients as V_i in $p - 1$ sets of cogredient variables.*

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* Loc. cit., p. 56.

† Loc. cit., p. 70.

‡ Compare Deruyts, loc. cit., p. 438.

Let $|\alpha|^\mu$ be the highest power of $|\alpha|$ that is contained in every element of the matrix φ. We denote the quotient $\phi_{ij}/|\alpha|^\mu$ by ψ_{ij}, and the result of substituting $x_i^{(j)}$ for α_{ij} in $|\alpha|$ by A, and the result of substituting $x_i^{(j)}$ for α_{ij} in ψ_{ij} by Y_{ij}. Then $X = A^\mu Y$, and $X' = A'^\mu Y'$, where A' and Y' are the same functions of the $x_i^{(j)\prime}$ that A and Y are of the $x_i^{(j)}$. From (7) we have $A = |\alpha| A'$. It follows that

$$\begin{aligned}\textstyle\sum_{k=1}^{k=t} K_k'Y_{ik}' &= A^{r-\mu}\textstyle\sum_{k=1}^{k=t} K_k'X_{ik}' = A^{r-\mu}\textstyle\sum_{j=1}^{j=t} K_jX_{ij}\\ &= A^\mu A^{r-\mu}\textstyle\sum_{j=1}^{j=t} K_j Y_{ij} = |\alpha|^\mu \textstyle\sum_{j=1}^{j=t} K_jY_{ij}.\end{aligned}$$

Hence in this case $\sum_{j=1}^{j=t} K_jY_{ij}$ is a relative covariant of weight μ for $i = 1, \cdots, t$. It will now be shown that for some i, $\sum_{k=1}^{k=t} K_kY_{ik}$ can be expressed in $p-1$ sets of cogredient variables. Since at least one function

$$Y_{ik}(x_1^{(1)}, \cdots, x_1^{(p)}, x_2^{(1)}, \cdots, x_2^{(p)}, \cdots, x_p^{(1)}, \cdots, x_p^{(p)})$$

does not contain A as a factor, and since A is irreducible, there is at least one set of constants c_{ij} such that $C = |c_{ij}| = 0$, and

$$Y_{ik}(c_{11}, \cdots, c_{1p}, c_{21}, \cdots, c_{2p}, \cdots, c_{p1}, \cdots, c_{pp}) \neq 0.$$

There is a linear relation with constant coefficients between the columns of C, say

$$\kappa_1c_{i1} + \kappa_2c_{i2} + \cdots + \kappa_pc_{ip} = 0, \quad (i = 1, \cdots, p),$$

where at least one κ, say κ_1, is different from zero. Hence

$$c_{i1} = \lambda_2c_{i2} + \lambda_3c_{i3} + \cdots + \lambda_pc_{ip}, \quad (i = 1, \cdots, p). \tag{8}$$

Therefore

$$Z_{ik} \equiv Z_{ik}(c_{12}, \cdots, c_{1p}, c_{22}, \cdots, c_{2p}, \cdots, c_{p2}, \cdots, c_{pp}) \neq 0,$$

where Z_{ik} is obtained from $Y_{ik}(c_{kl})$ by substitution (8).

Hence if we make the substitution

$$x_i^{(1)} = \lambda_2x_i^{(2)} + \lambda_3x_i^{(3)} + \cdots + \lambda_px_i^{(p)}, \quad (i = 1, \cdots, p) \tag{9}$$

upon Y_{ik}, we obtain

$$Z_{ik}(x_1^{(2)}, \cdots, x_1^{(p)}, x_2^{(2)}, \cdots, x_2^{(p)}, \cdots, x_p^{(2)}, \cdots, x_p^{(p)}) \not\equiv 0,$$

for in particular it is not zero when $x_i^{(j)} = c_{ij}$, $(i, j = 1, \cdots, p)$. Transformation (9) does not destroy the cogrediency of the variables. Since $Z_{ik} \not\equiv 0$, it follows that Z_{ik} for $k = 1, \cdots, t$ are linearly independent. For we have

$$\textstyle\sum_{k=1}^{k=t} K_kZ_{ik}A^\mu = \sum_{j=1}^{j=t} K_j'Z_{ij}'A'^\mu.$$

The polynomials $Z_{ik}A^{\mu}$ are transformed by the adjoint of φ, and according to the theorem of Schur mentioned above, a matrix which transforms a system of linearly dependent polynomials which are not all zero is reducible. Hence if the $Z_{ik}A^{\mu}$ were linearly dependent, the matrix φ would be reducible, contrary to our assumption.

5. *Conclusion.* We have proved the following theorem:

THEOREM. *If* $G_1, \cdots, G_h$ *are a system of polynomials in the* a_{ij}, *and* $G_1', \cdots, G_h'$ *the same functions of the* a_{ij}' *such that*

$$(G_1, \cdots, G_h) = (0, \cdots, 0)$$

is an invariantive property, then there exists a set of rational integral relative covariants $V_1, \cdots, V_v$ *in* $p-1$ *sets of cogredient variables such that* $(V_1, \cdots, V_v) = (0, \cdots, 0)$ *when and only when* $(G_1, \cdots, G_h) = (0, \cdots, 0)$.

PRINCETON UNIVERSITY

A CORRECTION

BY B. A. BERNSTEIN

In my paper in the November number of this BULLETIN (vol. 28, No. 8), the word *integers* should be replaced by the word *rationals* in line 16 of page 398 and in the table on page 399.

SHORTER NOTICES

Irrationalzahlen. By Oskar Perron. Berlin und Leipzig, Vereinigung Wissenschaftlicher Verleger, 1921. viii + 186 pp. Volume I of Göschen's Lehrbücherei, Gruppe I, Reine Mathematik.

Göschen's *Lehrbücherei* is to comprise works of textbook character, on topics chosen from the fields of mathematics, the exact sciences, and "Technik." They are intended primarily for students of universities and of *Technische Hochschulen.* Each volume is assumed to cover approximately the ground of a one semester university course.

The volume under consideration contains two parts of essentially different character. While the first half may be said to give a systematic treatment of the notion of an irrational number and its historical development, the last chapters are devoted to the interesting and not very widely known subject of the various methods of representing irrational numbers and their approximation by rational numbers and to a brief discussion of a certain class of transcendental numbers.

These last chapters, while not containing a large amount of new material, offer much that may be of interest even to professional mathematicians, particularly since the problem of approximation of irrational numbers is steadily gaining in importance and represents a difficult, but fascinating, field for research in which German and English mathematicians are making remarkable discoveries.

The book is divided into six chapters:

I. *The foundations* (32 pp.): Assuming the existence of rational numbers, the irrational numbers are defined by the Dedekind *Schnitt.* Refined questions of independence and consistency of the axioms employed are, in agreement with the character of the book, not dealt with.

II. *The notion of a limit* (28 pp.): The notions *upper and lower bound, limited, upper and lower limit* of a set of numbers are introduced. The *limit,* $\lim_{n=\infty} a_n$, is introduced only after the $\limsup_{n=\infty} a_n$, $\liminf_{n=\infty} a_n$ have been, in six pages, thoroughly established, and is defined by $\liminf a_n = \limsup a_n = \lim a_n$.—Convergent sequences and infinite series are briefly considered. The chapter ends with a four-page historical review of the various established methods of introducing irrational numbers (Cauchy, Bolzano, Weierstrass, Dedekind, Cantor, Méray, Bachmann, etc.) and a brief but clear discussion of their relations to one another.

III. *Powers and logarithms* (30 pp.): This chapter introduces powers with rational and irrational exponents, logarithms, the exponential series, etc., in the customary manner. It may be noted that the author, for formal reasons, assigns to 0^0 the value 1.

IV. *Various forms of representation of irrational numbers* (36 pp.): The following representations are explained and discussed (convergence, uniqueness, exceptional cases, etc.):

(*a*) $\gamma = \sum_{\nu=0}^{\infty} c_\nu/p^\nu$, p an arbitrary positive integer, c_ν integers, $0 \leqq c^\nu \leqq p-1$ for $\nu \geqq 1$, $c_\nu \leqq p-2$ for an infinite number of ν's (*systematic fractions*).

(*b*) $\gamma = [b_0, b_1, b_2, \cdots]$, b_ν positive integers (*continued fractions*); periodicity, Lagrange's theorem, Lambert's fraction for $(e+1)/(e-1)$, Euler's fraction for e, etc.

(*c*) $\gamma = c_0 + \sum_{\nu=1}^{\infty} c_\nu/(p_1, p_2, \cdots, p_\nu)$, p_ν arbitrary positive integers, c_ν integers, $0 \leqq c_\nu \leqq p_\nu - 1$ for $\nu \geqq 1$, $c_\nu \leqq p_\nu - 2$ for an infinite number of ν's (*Cantor's series and Cantor's algorithm*). Irrational when infinite number of terms.

(*d*) $$\gamma = c + \frac{1}{q_1} + \sum_{\nu=1}^{\infty} \frac{1}{(q_1-1)q_1(q_2-1)q_2\cdots(q_\nu-1)q_\nu} \cdot \frac{1}{q_{\nu+1}},$$

c and q_ν integers, $q_\nu \geqq 2$ for $\nu \geqq 1$ (*Lüroth's series*). Irrational except when $q_1, q_2, q_3, \cdots$ form a periodic sequence.

(*e*) $\gamma = c + \sum_{\nu=0}^{\infty} (q_1 q_2 \cdots q_{\nu+1})^{-1}$, c, q_ν integers, $q_1 \geqq 2$, $q_{\nu+1} \geqq q_\nu$ (*Engel's series of the first kind*). Irrational except when, from a certain ν on, always $q_{\nu+1} = q_\nu$.

(*f*) $\gamma = c + \sum_{\nu=0}^{\infty} (q_{\nu+1})^{-1}$, c, q_ν integers, $q_1 \geqq 2$, $q_{\nu+1} \geqq (q_\nu - 1)q_\nu + 1$ (*Engel's series of the second kind*). Irrational except when for all sufficiently large ν, $q_{\nu+1} = (q_\nu - 1)q_\nu + 1$.

(*g*) $\gamma = \prod_{\nu=0}^{\infty}\left(1 + \frac{1}{q_\nu}\right)$, q_ν positive integers, $q_{\nu+1} \geqq q_\nu^2$ and not all $q_\nu = 1$ (*Cantor's products*). Irrational except when from a certain ν on $q_{\nu+1} = q^2$. Cantor's products are based on the identity

$$(1-x)\cdot\prod_{\nu=0}^{n-1}(1 + x^{(2^\nu)}) = 1 - x^{(2^n)}.$$

The reviewer does not know of any other place where these different methods of representing irrational numbers have been collected and discussed.

V. *Approximation to irrational numbers by means of rational numbers* (30 pp.): This chapter offers a satisfactory elementary introduction to the theory of *rational* and *diophantine approximations*, which is constantly growing in importance. Starting with the well known approximation to an irrational number ξ, $|\xi - p/q| < q^{-2}$ *for an infinite number of fractions* p/q, and from Hurwitz' complementary theorems on this inequality, the author advances to the proof and discussion of the following theorem concerning the simultaneous approximation of any finite number of given numbers $\xi_1, \cdots, \xi_n$: $|\xi_\nu - p_\nu/q| < q^{-1-1/n}$ *for an infinite number of* p_ν/q, *provided the* ξ_ν *are not all rational.*

After considering more complicated systems, a deep-lying theorem of Kronecker* concerning the simultaneous approximation of a system of linear functions of any number of variables is derived.

The author claims a considerable simplification as compared with Kronecker's proof. The Kronecker theorem has found within the last years unexpected applications, for example in the theory of Riemann's Zeta-function.

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* Kronecker, *Näherungsweise ganzzahlige Auflösung linearer Gleichungen*, SITZUNGSBERICHTE DER PREUSSISCHEN AKADEMIE ZU BERLIN, 1884, p. 1179 ff. The complete theorem is too long and complicated to be quoted here.

VI. *Algebraic and transcendental numbers* (24 pp.): Besides the fundamental definitions and theorems (including the proof that e and π are transcendental numbers) a certain class of transcendental numbers which have been called by Maillet* *Liouville numbers* are studied. Since Maillet introduced the name and made in his book a systematic (although somewhat obscure) study of these numbers and since Perron in another work (*Die Lehre von den Kettenbrüchen*, Leipzig, 1913) gives full credit to Maillet, it is obviously an oversight that Maillet's book is not mentioned in the *Irrationalzahlen.*

The literature references are arranged for each chapter separately and seem fairly complete. However, Minkowski, *Diophantische Approximationen*, Leipzig, 1907, is not quoted. Borel, *Leçons sur la Théorie de la Croissance*, Paris, 1910, pp. 118–168, might have been mentioned in connection with chapters V, VI. It is not very satisfactory that even in the case of large books no page reference is given; a reference such as: L. Euler, *Introductio in Analysin infinitorum*, I, 1748 (a book of over 300 quarto pages, in Latin), is perhaps not easily run down.

The only American author referred to seems to be Huntington (TRANSACTIONS OF THIS SOCIETY, vol. 6 (1905).

A. J. KEMPNER

Grundlehren der Neueren Zahlentheorie. By Paul Bachmann. Volume III of Göschen's Lehrbücherei, Gruppe I, Reine Mathematik. Berlin und Leipzig, Vereinigung Wissenschaftlicher Verleger, 1921. xi + 252 pp.

The book under discussion represents the second edition of volume LIII of the well known Sammlung Schubert, G. J. Göschensche Verlagshandlung, Leipzig, 1907. No important changes have been made. New is a chronological table of the known proofs of the famous *Law of Quadratic Reciprocity:* fifty-six proofs from the year 1796 (Gauss' first proof, published 1801) to Frobenius' modification of Zeller's proof, 1914.† The mathematical basis of each proof is indicated; in thirty-two cases Gauss' lemma or a variant of Gauss' lemma is given as the foundation. A five-page alphabetic index has also been added.

Since the first edition was reviewed by J. W. Young in this BULLETIN (vol. 15 (1908–9), pp. 463-5), it is not necessary to consider in detail the mathematical contents of this excellent little book. The small corrections which were suggested in this review have been carried out.

The new edition is posthumous; it contains a five-page necrology, *Zum Gedächtnisse von Paul Bachmann*, by Robert Haussner. Bachmann's influence in stimulating interest in the theory of numbers has been so great that American readers may be interested in a few notes concerning his life and his work.

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* Maillet, *Introduction à la Théorie des Nombres Transcendants,* Paris, 1906, particularly chapters II, III.

† A less complete table is contained in Bachmann's *Niedere Zahlentheorie*, vol. I, p. 203 ff.

Paul Bachmann, born Berlin, June 22, 1837; studied mathematics 1855–1862 at Göttingen and Berlin (at Göttingen, together with Dedekind, under Dirichlet); Privatdozent at Breslau, 1864; ausserordentlicher Professor, Breslau, 1867; Professor at the Akademische Lehranstalt, Münster i. W., 1873; retired 1890; did not occupy an official position from this time on until his death, March 31, 1920.

Bachmann's influence is based, not so much on his original contributions to mathematics,* as on his series of scientific textbooks on the theory of numbers. It was his ambition to cover in these books the total field of this theory.

The *Gesamtdarstellung* included, up to the time of Bachmann's death, the following volumes:

I. *Elemente der Zahlentheorie*, 1892,
II. *Analytische Zahlentheorie*, 1894,
III. *Lehre von der Kreisteilung*, 1872,
IV (1). *Arithmetik der quadratischen Formen*, 1898,
IV (2). (*Über Reduktion der Formen*),
V. *Arithmetik der Zahlenkörper*, 1905.

Of these, IV (2) was finished in manuscript in 1915, but could not, on account of economic conditions in Germany, be published, although "nach dem übereinstimmenden Urteile aller Mathematiker, die das Manuscript kennen gelernt haben, ein ganz ausgezeichnetes Werk vorliegt." For a second volume of V, *Über spezielle Zahlenkörper*, only preliminary work was carried out.

Besides the *Gesamtdarstellung*, Bachmann has published several books:
Vorlesungen über die Natur der Irrationalzahlen, 1892,
Niedere Zahlentheorie, vol. I, 1902,
Niedere Zahlentheorie, vol. II, 1910 (Additive Zahlentheorie),
Grundlehren der Neueren Zahlentheorie, 1907 (2d ed. 1921),
Das Fermatproblem in seiner bisherigen Entwicklung, 1919.

The difficulty of carrying out the program outlined by this list will be realized by every mathematician familiar with the theory of numbers. With admirable skill Bachmann brings out underlying unifying principles and, even in such fields as the additive theory of numbers, traces the threads connecting a large number of apparently disconnected results.

Since many of the books—for example those dealing with the analytic theory and with the additive theory—were written at a period when these branches of mathematics were in a stage of vigorous development, it is inevitable that some of the volumes no longer adequately represent the complete theory; but for a long time to come it will be safe to advise any student interested in the theory of numbers to study carefully Bachmann's works. Perhaps the highest praise one can give these books is to emphasize that the progress in mathematics which has made a few of them appear slightly antiquated is probably in no small measure due to the interest aroused by them.

A. J. Kempner

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* This statement should not be interpreted as a criticism of the value of his original contributions.

Étude Géométrique des Transformations Birationnelles et des Courbes Planes. By Henri Malet. Paris, Gauthier-Villars et Cie., 1921. viii + 259 pp.

About half of this book is devoted to the study of plane curves and the linear transformation, the remainder to the quadratic and the general birational transformation. According to the preface it is the work of an engineer—the product of leisure hours at the front. The writer appears to be acquainted with the works of the older French geometers but entirely out of touch with modern developments. Only a score of references to some twelve mathematicians were noted. Of these Cremona is the most recent. The names of Cayley, Noether, Clebsch, Bertini, S. Kantor do not appear, nor is any present-day treatise mentioned.

The treatment, characterized as geometric, is based neither on a system of axioms nor on a set of theorems proved analytically. It is rather descriptive and intuitional. Though the book may be useful as collateral reading, it is not a safe guide for the novice. For example, the last two theorems (pp. 250, 255), which state that any two birationally related curves can be transformed the one into the other by a birational transformation of the plane (a Cremona transformation), are not correct.

ARTHUR B. COBLE

The Fourth Dimension. By E. H. Neville. Cambridge, University Press, 1921. 56 pp.

This little book is intended by the author to serve as an introduction for the general reader to the notions of four-dimensional space, this topic having become of interest to large numbers of laymen on account of the wide general interest in the work of Einstein and the theory of relativity.

The author presupposes the reader to be familiar only with elementary trigonometry and with the solution of simultaneous linear equations in algebra and, while literally his treatment justifies this supposition, a considerable knowledge of linear equations and of linear dependence is necessary as well as a considerable mathematical maturity. The treatment is purely analytic and constitutes indeed a construction of four-dimensional analytic space on the foundation of the real number system. The book is written in the spirit of mathematical foundations and is remarkable for its rigor and precision of statement. It is difficult for the reviewer to see how the book can be of interest to any large class of readers in this country. It is too abstract and presupposes too much mathematical maturity for the lay reader; on the other hand, individuals having sufficient mathematical training to appreciate the author's exposition are very likely not to need it. The book seems to the reviewer of interest primarily as giving the details necessary for constructing an abstract space on a purely analytic foundation. The introduction of the ideas of direction and measurement of angles in this connection is of special interest. As a popular introduction to the terms and notions used in the discussion of the theory of relativity, it can be of little use for the reasons already mentioned.

J. W. YOUNG

An Introduction to Electrodynamics. By Leigh Page. Boston, Ginn and Company, 1922. 6 + 134 pp.

This book fills a gap between more advanced mathematical treatments and the purely physical expositions. It will be found very valuable on this account and because of its conciseness. The reviewer is sure that it would form a solid basis for an advanced course in mathematical physics.

Nevertheless there are certain criticisms which depend, perhaps, on mere personal taste. The author objects to the usual introduction through the laws of Coulomb, Ampère, and Faraday. Surely these laws are not of mere historical interest. They also constitute the foundation of the concepts of electrodynamics.

Professor Page has preferred to base his theory on the existence of electrons, on the principle of relativity, and on the hypothesis of "elements" moving with the speed of light throughout the space surrounding each individual electron. There is not room for a full reasoned criticism of this last hypothesis—as a basis for an introductory textbook. But in the present uncertain state of electric and dynamic theory it is better to keep in touch as far as possible with well-established experimental results.

The discussion on page 10 of related and ideal reference systems is not at all clear and appears to lack a little in rigor.

At the end of Chapter 4 the radius of the negative electron is asserted to be 1.88×10^{-13} cms. as if it did not depend on special assumptions with regard to the structure of which we know exactly nothing.

An index would have made the book more useful.

P. J. Daniell

Wahrscheinlichkeitsrechnung und ihre Anwendung auf Fehlerausgleichung, Statistik, und Lebensversicherung. By Emanuel Czuber. Vol. II, Third Edition. Leipzig and Berlin, Teubner, 1921. x + 470 pp.

Die statistichen Forschungsmethoden. By Emanuel Czuber. Wien, L. W. Seidel und Sohn, 1921. x + 238 pp.

The first of these books may be properly regarded as a reprint of the second edition (1910) with minor changes including a renumbering of the articles to make Volume II a continuation of the revised Volume I.

The second book seems to have been inspired largely by Yule's contributions to statistical theory. In fact, Yule's theory of the statistics of attributes, including the notation, is given in the first 34 pages of the book, and the outline of the entire book follows closely Yule's *Introduction to the Theory of Statistics.* Features of the book that appeal to the reviewer particularly are the inclusion of the summation method of computing the arithmetic mean and standard deviation, the derivation of the Poisson exponential limit in dealing with the problem of small probabilities, the treatment of the stability of percentiles, and the effective presentation of concrete illustrations. The treatment of correlation theory follows Yule closely. Throughout the book, the exposition of theory seems especially clear; but, on account of the fact that the development of theory follows Yule so closely, the book will be useful to the American student and teacher of statistics mainly as a source of concrete illustrations.

H. L. Rietz

Der Kreisel. By R. Grammel. Braunschweig, Vieweg and Son, 1920. 10 + 350 pp.

The book opens with vector analysis and the mechanics of rigid bodies. The rotating body under no forces is handled with clarity and special consideration to physical concepts. There is then taken up the problem of a rotating body whose axis is forced to move along a given path. The 'gyroscopic couple' which is required in this case is used continually in the discussions in the remainder of the book. There are plentiful applications—to vehicles, ships, the gyroscopic compass and other stabilizing gyroscopes. The hanging and the stabilized monorail are both considered and the treatment of the aeroplane is full. There is even some reference to gyroscopic action in atoms.

Dr. Grammel believes that a reform is necessary to replace long-winded phrases by simple words, to correspond to the simplicity of the ideas expressed. There is room for an interesting discussion among American mathematical engineers on this topic; but the modern man uses many long technical terms quite naturally. Perhaps 'moment of momentum' and 'moment of inertia' might be discarded with advantage.

The book is to be highly recommended for its attention to basic principles. In studying the bicycle, for example, the theory of a riderless mechanism is omitted because it has no value. It is replaced by a statement as to the general mechanical properties. The book does not take the place of a complete mathematical treatise on rotating systems, but there is no diffidence in the use of mathematics as a hand-maid. The illustrations are striking and a valuable engineering course might be given with this book as a foundation.

P. J. Daniell

Mathematica Delectans. By Dr. G. Kowalewski. Heft I. *Boss Puzzle und Verwandte Spiele.* Leipzig, Wilhelm Engelmann, 1921. 72 pp.

This little book is the first of a projected series on mathematical recreations and games which Professor Kowalewski proposes to publish. This first volume is concerned with the so-called boss puzzle or what is more familiarly known in this country as the fifteen puzzle. He finds that all possible configurations fall into two equal classes and that all the configurations of one class are equivalent under the allowable motions among themselves, but that no configuration of one class is equivalent to any configuration of the other. The latter part of the booklet is devoted to various generalizations of the boss puzzle in which the ideas previously introduced are effectively applied. The book is so elementary in character that it would hardly justify a notice in the Bulletin were it not for the fact that the abstract ideas involved are of very general application and wide interest, especially as exhibited in the discussion towards the end of the booklet. The succeeding volumes of the proposed series will be awaited with much interest. The author has struck a new note and, I feel inclined to say, established a new standard regarding the treatment of problems ordinarily held to be outside the field of serious mathematical endeavor.

J. W. Young

NOTES

The third number of volume 22 (1921) of the TRANSACTIONS OF THIS SOCIETY contains the following papers: *Polynomials and their residue systems (continued)*, by A. J. Kempner; *On certain two-point properties of general families of curves*, by J. Douglas; *Properties of the solutions of certain functional differential equations*, by W. B. Fite; *Note on a class of polynomials of approximation*, by D. Jackson; *On certain numerical invariants of algebraic varieties with application to Abelian varieties*, by S. Lefschetz.

The second number of volume 23 (1922) of the TRANSACTIONS OF THIS SOCIETY contains the following papers: *Differential geometry of the complex plane*, by J. L. Coolidge; *Invariantive characterizations of linear algebras with the associative law not assumed*, by C. C. MacDuffee; *Curves invariant under point-transformations of special type*, by Mary F. Curtis; *Die Zerlegung von Primzahlen in algebraischen Zahlkörpern*, by Andreas Speiser; *The elliptic modular functions associated with the elliptic norm curve* E^7, by Roscoe Woods; *Linear equations with two parameters*, by Anna J. Pell; *The theory of functions of one Boolean variable*, by Karl Schmidt.

On account of the reduction of stock due to unusual sales, the previously announced reduced prices on the publications of this Society have been withdrawn. A new schedule of prices will appear in the February issue.

Professor C. Segre, of the board of editors of the ANNALI DI MATEMATICA, PURA ED APPLICATA, has written to Professor Virgil Snyder expressing the thanks of the board for the cash contributions and the new subscriptions received through the cooperation of the American Mathematical Society. These subscriptions will begin with volume 32 of the third series, the first number of which will be issued soon.

At a meeting of the Executive Committee of the Division of Physical Sciences of the National Research Council, in Boston on December 26, 1922, Professor Oswald Veblen was elected Vice-Chairman of the Division for the year 1922–23.

Sir Ernest Rutherford has been elected president of the British Association for the Advancement of Science, in succession to Sir C. S. Sherrington. The Association will meet at Liverpool in the summer of 1923.

At the meeting of scientists held at Leipzig in September, 1922, preliminary steps were taken to found a new society for the advancement of the theoretical engineering sciences, including mechanics and applied mathematics. A committee was formed, with Professor L. Prandtl, of Göttingen, as chairman, to complete the organization, and to propose a suitable name for the society.

The Jablonowski Society has awarded its prize for a memoir on addition theorems (see this BULLETIN, vol. 26, p. 332) to Dr. P. J. Myrberg, of the University of Helsingfors.

The firm of Ulrico Hoepli, in Milan, has recently printed a chronological catalogue of its publications from 1872 to 1922, in commemoration of the fiftieth anniversary of its foundation. This house has published, besides the *Manuali Hoepli*, the collected works of Beltrami, Betti, Brioschi, and Cremona.

Mr. Walter Denston, formerly of the Imperial Naval Engineering College, Kronstadt, Russia, has been appointed assistant professor of mathematics at Kenyon College.

Associate Professor Mary E. Sinclair, of Oberlin College, is on leave of absence during the second semester of 1922–23 as holder of the Pratt Fellowship of the National Association of University Women, and is studying at the University of Chicago.

At Purdue University, Mr. W. H. Lyons has been appointed instructor in mathematics.

At the University of Michigan, Professor W. H. Butts has retired after serving twenty-four years as instructor and professor of mathematics and fourteen years as assistant dean of the Colleges of Engineering and Architecture. He has been appointed professor emeritus. Mr. Clair Reid, instructor at Purdue University, has been appointed instructor.

Dr. C. C. Wylie is associate in astronomy and acting head of the department at the University of Illinois.

Dr. C. R. Adams of Harvard University has been appointed instructor in mathematics at Brown University for the year 1923–1924.

Mr. Paul Bruck, Astronomer Adjoint at the observatory of Besançon, died July 31, 1922, at the age of sixty-six years.

Professor H. F. Buchholz, of the University of Halle, died November 24, 1921, at the age of fifty-five years.

Professor Jacob Rosanes, of the University of Breslau, died January 7, 1922, at the age of seventy-nine years.

Lieut. Col. P. J. B. H. Brocard, well known for his work in the geometry of the triangle, died January 16, 1922, at the age of seventy-six years.

Professor A. N. Favaro, of the University of Padua, died September 30, 1922, at the age of seventy-five years.

Dr. C. G. Knott, reader in applied mathematics at the University of Edinburgh, died October 26, 1922, at the age of sixty-six years.

Professor Frederick Anderegg, of Oberlin College, died October 9, 1922, at the age of seventy years.

Professor S. S. Keller, head of the department of mathematics at the Carnegie Institute of Technology, died January 12, 1923.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

Bassett (R.). Dalton plan assignments. II: Mathematics and science. London, Bell, 1922. 64 pp.

Breitensteins Repetitorium. Repetitorium der höheren Mathematik. Erster Teil: Differentialrechnung. Von F. Huneke. 2te neuverfasste Auflage. Leipzig, Barth, 1922. 6 + 160 pp.

Dintzl (E.) und Vaselli (C.). Aufgaben aus der reinen und angewandten Mathematik. 1ter Band. Wien, Gerold, 1922.

Fazzari (G.). Elementi di aritmetica con note storiche e numerose questioni varie per le scuole medie superiori. 6a edizione. Palermo, Trimarchi, 1923. 196 pp.

Huneke (F.). See Breitensteins Repetitorium.

Kniebes (W.). Ueber das Hypersoma. (Diss.) Bonn, 1921.

Kraïtchik (M.). Théorie des nombres. Avec une préface de M. d'Ocagne. Paris, Gauthier-Villars, 1922. 10 + 229 pp.

Ling (G. H.), Wentworth (G.), and Smith (D. E.). Elements of projective geometry. Boston, Ginn, 1922. 6 + 186 pp. $2.80

Montel (P.). See Vessiot (E.).

Myrberg (L.). Ueber Systeme analytischer Funktionen, welche ein Additions theorem besitzen. (Preisschriften gekrönt und herausgegeben von der Fürstlich Jablonowskischen Gesellschaft zu Leipzig.) Leipzig, Teubner, 1922. 24 pp.

Neubauer (E.). Globoidschneckenlinien, globoidische Strahlensysteme und Regelflächen. (Diss., Halle-Wittenberg.) Halle, 1922.

d'Ocagne (M.). See Kraïtchik (M.).

Pearson (K). Tables of the incomplete Γ-function. Edited by Karl Pearson. London, His Majesty's Stationery Office, 1922. 4to. 32 + 164 pp.

Picard (E.). Discours et mélanges. Paris, Gauthier-Villars, 1922. 5 + 292 pp.

Smith (D. E.). See Ling (G. H.).

Smith (H. B.). Foundations of formal logic. Philadelphia, University of Pennsylvania Press, 1922. 8vo. 56 pp.

Tonelli (L.). Fondamenti di calcolo delle variazioni. Volume 1. Bologna, Zanichelli, 1921. 8vo. 466 pp.

Vaselli (C.). See Dintzl (E.).

Vessiot (E.) et Montel (P.). Cours de mathématiques générales, professé à la Faculté des Sciences de Paris en 1919–1920. Paris, Librairie de l'Enseignement technique, 1921. 8vo. 504 + 582 pp.

Wentworth (G.). See Ling (G. H.).

Whitehead (A. N.). The rhythm of education. London, Christophers, 1922. 30 pp.

Woods (F. S.). Higher geometry. An introduction to advanced methods in analytic geometry. Boston, Ginn, 1922. 10 + 423 pp. $5.00

PART II. APPLIED MATHEMATICS

ACHALME (—.). Les édifices physico-chimiques. Tome 1: L'atome, sa structure, sa forme. Tome 2: La molécule. Equilibres et réactions chimiques. Paris, Payot, 1921–1922. 246 + 232 pp.

AIGNER (E.). Unterwasserschalltechnik. Grundlagen, Ziele, und Grenzen. (Submarine Akustik in Theorie und Praxis.) Berlin, M. Krayn, 1922. 8vo. 332 pp.

ALLIATA (G.). Verstand contra Relativität. Zum Nachweis der Translation des Sonnensystems. Leipzig, Hillmann, 1922. 8vo. 83 pp.

ALIOTTA (A.). Teoria d'Einstein. Le mutevoli prospettive del mondo. Palermo, R. Sandron, 1922. 8vo. 130 pp.

ANDREWS (E. S.). An introduction to applied mechanics. Cambridge, 1920.

AUBERT (P.) et PAPELIER (G.). Exercices de mécanique. 2e édition. Paris, Vuibert, 1921. 8vo. 356 pp.

AUTENRIETH (E.). Technische Mechanik. 3te Verbesserte Auflage. Neubearbeitet von M. Ensslin. Berlin, Springer, 1922. 15 + 564 pp.

BANGERT (K.). Masse und Masssysteme mit besonderer Berücksichtigung der Elektrotechnik. Leipzig, Kesselringsche Hofbuchhandlung, 1922. 8vo. 8 + 110 pp.

BARKER (T. V.). Graphical and tabular methods in crystallography as the foundation for a new system of practice. London, T. Murby, 1922. 16 + 152 pp.

BAYLISS (W. M.). Life and the laws of thermodynamics. London, Oxford University Press, 1922. 8vo. 12 pp.

BEDELL (F.). The airplane. New York, Van Nostrand, 1920. 257 pp. $3.00

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BINGHAM (E. C.). Fluidity and plasticity. New York and London, Macmillan, 1922. 11 + 440 pp.

BLOCH (L.). Le principe de la relativité et la théorie d'Einstein. Paris, Gauthier-Villars, 1922. 3 + 42 pp.

BOHR (N.). Drei Aufsätze über Spektren und Atombau. Leipzig, Vieweg, 1922. 8vo. 7 + 148 pp.

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BOULVIN (J.). Cours de mécanique appliquée aux machines. 6e volume: Machines et chaudières marines et leurs appareils auxilières. 8e volume: Compresseurs, transmission du travail à distance, appareils de levage. Paris, Michel, 1921.

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Lewis (E. P.). See Duff (A. W.).

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Müller (J. J. C.). See Christiansen (C.).

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Scherrer (F. R.). See Euler (L.).

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AN UNCOUNTABLE, CLOSED, AND NON-DENSE POINT SET EACH OF WHOSE COMPLEMENTARY INTERVALS ABUTS ON ANOTHER ONE AT EACH OF ITS ENDS*

BY R. L. MOORE

On page 92 of the 1907 edition of Hobson's *The Theory of Functions of a Real Variable,* and again on page 113 of the second edition of the same treatise, there occurs the following statement: †

"*A non-dense closed set is enumerable if its complementary intervals are such that every one of them abuts on another one at each of its ends.*"

To prove this statement, Hobson lets G denote the non-dense closed set in question and argues, in part, as follows:

"In this case, all the points of G are either end-points of adjacent intervals, or limiting points, on both sides of a sequence of such end-points; unless a ‡ or b ‡ be a limiting point, in which case it belongs to G. The end-points have the same cardinal number as the rational numbers, since the set of intervals is enumerable. Moreover the external § points form a finite set, or an enumerable set; because to each such external point there corresponds an enumerable set of end-points of which it is the limiting point and in this correspondence *any one end-point can correspond to at most two limiting points, one on each side of it.*"

Just what is the meaning of the above italicized statement? If P is an external point, then, by definition, there exists a sequence of end-points (of intervals complementary to G) such that P is the sequential limiting point of this sequence. But clearly there exist infinitely many such sequences and if X is an end-point of any interval whatsoever complementary to G then there is one sequence of such end-points which contains X and has P as its limiting point. In the correspondence imagined by Hobson does P correspond, then, to every such point X? If not, then apparently it is intended that for each point P there should be selected, in some manner or other, some particular sequence of end-points converging to P and that the points of this particular sequence only should be considered as corresponding to P. Even if this be done, however, what justification (if any) is there for the statement that, in the correspondence so established, any one end-point can correspond to at most two limit points, one on each side of it? Certainly it is possible that for three different external points P_1, P_2, P_3 there should be selected three sequences $P_{11}, P_{12}, P_{13}, \cdots; P_{21}, P_{22}, P_{23}, \cdots; P_{31}, P_{32}, P_{33}, \cdots$ such that (1) these sequences converge respectively to P_1, P_2, P_3, and their points correspond respectively to the points P_1, P_2, P_3, and (2) $P_{11} = P_{21} = P_{31}$. In this case the end-point P_{11} would correspond to three different limiting points, contrary to Hobson's statement. In fact there exists a non-dense closed point set G satisfying the hypothesis of Hobson's proposition and such that there is no way whatsoever of establishing the correspondence indicated by Hobson without having some end-point correspond to uncountably many limiting points. To see this, let K denote a non-dense perfect set of points lying on an interval AB and containing the points A and B. Consider the intervals A_1B_1, A_2B_2, A_3B_3, $\cdots$ which are complementary to K (with respect to AB). For every two positive integers n and m let P_{nm} and $\overline{P}_{nm}$ denote points between A_n and B_n situated so that $P_{nm}A_n = \overline{P}_{nm}B_n = A_nB_n/2m$. Let G denote the point set obtained by adding, to the point set K, all the P_{nm}'s and all the $\overline{P}_{nm}$'s.

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* Presented to the Society, December 29, 1922.

† This statement will be called Proposition A.

‡ Here a and b apparently denote the end-points of some interval which contains the set G.

§ External points are defined by Hobson as points such as are not end-points of any contiguous interval but are limit points on both sides of such end-points.

The existence of this example disproves Proposition A.

THE UNIVERSITY OF TEXAS

TOTAL GEODESIC CURVATURE AND GEODESIC TORSION*

BY J. K. WHITTEMORE

In a paper on the gyroscope, presented to this Society April 23, 1921, Professor W. F. Osgood introduced the notion of the "bending" of a spherical curve. The "bending" is defined as the rate of turning, per unit length of arc of the curve, of the plane determined by the tangent to the curve and the normal to the surface. It is the purpose of this note to show that the bending of a curve on any surface is equal to

$$\sqrt{\frac{1}{\rho_g{}^2}+\frac{1}{\tau_g{}^2}},$$

where ρ_g and τ_g are the radii of geodesic curvature and geodesic torsion respectively. An expression, believed to be new, for the geodesic torsion of any curve of the surface is also derived. Since the rate of turning of the principal normal of a curve,

$$\sqrt{\frac{1}{\rho^2}+\frac{1}{\tau^2}},$$

where ρ and τ are the radii of curvature and torsion respectively, is called the total curvature of a curve,† it seems appropriate to replace the term "bending" by "total geodesic curvature."

Let Γ be any curve of a surface S and P be any point of Γ; let $\bar{\omega}$ be the angle measured from the positive direction of the principal normal to Γ at P to the positive direction of the normal to S at P, the angle being measured toward the positive binormal; let the direction cosines of the positive direction of the normal to S be X, Y, Z, and those of the positive directions of principal normal and binormal be respectively l, m, n and λ, μ, ν. Then the direction cosines of the normal to the plane of the tangent to Γ and the normal to S are

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[*] Presented to the Society, October 29, 1921.

[†] ENCYCLOPÄDIE DER MATHEMATISCHEN WISSENSCHAFTEN, III, 3, 1, p. 82.

$$\begin{aligned} A &= l \sin \bar{\omega} - \lambda \cos \bar{\omega}, \\ B &= m \sin \bar{\omega} - \mu \cos \bar{\omega}, \\ C &= n \sin \bar{\omega} - \nu \cos \bar{\omega}, \end{aligned}$$

and the total geodesic curvature, g, is given by

$$g^2 = \Sigma \left(\frac{dA}{ds} \right)^2 = \frac{\sin^2 \bar{\omega}}{\rho^2} + \left(\frac{1}{\tau} - \frac{d\bar{\omega}}{ds} \right)^2 = \frac{1}{\rho_g{}^2} + \frac{1}{\tau_g{}^2},$$

where s is the arc of Γ. The third member of the preceding equation is obtained by differentiating the values given for A, B, C and using the Frenet-Serret formulas.*

It is of interest to note the value of the total geodesic curvature for certain special curves of a surface. 1. The total geodesic curvature is numerically equal to the geodesic curvature when the geodesic torsion vanishes, that is, when Γ is tangent at P to a line of curvature of S; this occurs at all points of Γ when and only when Γ is a line of curvature; it is true in particular for all spherical curves, the case considered by Professor Osgood. 2. The total geodesic curvature is numerically equal to the geodesic torsion when the geodesic curvature vanishes, that is, when Γ osculates a geodesic line at P; this occurs at all points of Γ when and only when Γ is a geodesic line; for such a line total geodesic curvature, geodesic torsion and torsion are numerically equal. The total geodesic curvature vanishes for all points of a geodesic line of curvature, which is necessarily a plane curve. 3. If Γ is an asymptotic line, the total geodesic curvature is at every point equal to the total curvature. This fact is evident from the definitions of the two curvatures, and appears also from the equations, true for any asymptotic line,

$$\rho = \pm \rho_g, \qquad \tau = \tau_g.$$

There exist, however, on every surface curves other than the asymptotic lines such that at every point the total geodesic curvature is equal to the total curvature. Along such a curve the following equation must be satisfied:

$$\left(\frac{d\bar{\omega}}{ds} \right)^2 - \frac{2}{\tau} \frac{d\bar{\omega}}{ds} = \frac{\cos^2 \bar{\omega}}{\rho^2}.$$

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* See Eisenhart, *Differential Geometry*, pp. 17, 132, 138.

The asymptotic lines are given by the solution, $\cos \bar{\omega} = 0$. For a given surface with given parameters, u, v, this equation becomes a differential equation of the third order for v as a function of u. If on the other hand any curve is given, a "surface band," that is, the curve and the normal to a surface at each point of the curve, is determined by a solution $\bar{\omega}$ of this equation such that the given curve has the required property on any surface containing the band. If, in particular, the given curve is plane, the equation gives for the surface band $\cos \bar{\omega} = \operatorname{sech} \varphi$ where φ is the angle made by the tangent to the given plane curve with any fixed direction in the plane.

We now prove that the geodesic torsion of any curve Γ of S is given by

$$\frac{1}{\tau_g} = \sin \theta \frac{d\sigma}{ds},$$

where σ is the arc of the curve corresponding to Γ in the spherical representation of S and θ is the angle measured from the positive direction of Γ to the positive direction of the corresponding curve in the spherical representation, the angle being measured in the direction of rotation from the positive direction of the parametric curve (v), or v constant, on S to the positive direction of (u).

The direction cosines of the positive directions on Γ and on its spherical representation are respectively dx/ds, dy/ds, dz/ds and $dX/d\sigma$, $dY/d\sigma$, $dZ/d\sigma$. We find

$$\begin{vmatrix} \dfrac{dx}{ds} & \dfrac{dy}{ds} & \dfrac{dz}{ds} \\ \dfrac{dX}{d\sigma} & \dfrac{dY}{d\sigma} & \dfrac{dZ}{d\sigma} \\ X & Y & Z \end{vmatrix} = \sin \theta.$$

Suppose the parametric system to be such that Γ is a curve (v) and that the system is orthogonal so that $F = 0$. We may then evaluate the determinant above, writing

$$\frac{dx}{ds} = \frac{1}{\sqrt{E}} \frac{\partial x}{\partial u}, \qquad \frac{dX}{d\sigma} = \frac{ds}{d\sigma} \frac{1}{\sqrt{E}} \frac{\partial X}{\partial u},$$

using the formula given by Eisenhart[*] for $\partial X/\partial u$, replacing $-D'/H$ by $1/\tau_g$, and noting that

$$\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial y}{\partial u} & \frac{\partial z}{\partial u} \\ \frac{\partial x}{\partial v} & \frac{\partial y}{\partial v} & \frac{\partial z}{\partial v} \\ X & Y & Z \end{vmatrix} = H.$$

We find

$$\frac{1}{\tau_g}\frac{ds}{d\sigma} = \sin\theta,$$

so proving the formula given. A somewhat simpler proof can be given for the numerical correctness of this result, that is, if the question of sign is disregarded. Certain properties of geodesic torsion appear clearly from the preceding formula. 1. If Γ is tangent to a line of curvature of S at P, the tangents to Γ at P and to the spherical representation of Γ at the corresponding point are parallel, and $\theta = 1/\tau_g = 0$. 2. If Γ is tangent to an asymptotic line of S at P, we have $\theta = \pm\pi/2$ and $1/\tau_g = \pm d\sigma/ds$. Since in the direction of an asymptotic line $d\sigma^2/ds^2 = -K$, where K is the total curvature of S, we have Enneper's theorem:[†] in the direction of an asymptotic line $1/\tau_g = \pm\sqrt{-K}$. 3. Since the spherical representation of a minimal surface is conformal,[‡] $1/\tau_g = \pm\sqrt{-K}\sin 2\psi$ for any curve on such a surface, where ψ is the angle of the tangent to the curve and the tangent to either line of curvature. If a minimal surface is applicable to a surface of revolution, every geodesic line of the minimal surface, which in the application corresponds to a meridian of the surface of revolution, cuts all the lines of curvature of one family at the same angle.[§] For such a geodesic the torsion varies as $\sqrt{-K}$ since ψ is constant.

YALE UNIVERSITY

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[*] Loc. cit., pp. 116, 138.

[†] Eisenhart, loc. cit., pp. 140, 141.

[‡] Loc. cit., p. 251.

[§] This result was first given by E. Bour, *Théorie de la déformation des surfaces*, JOURNAL DE L'ÉCOLE POLYTECHNIQUE, vol. 39 (1862).

THE NAME "DIVERGENT" SERIES

BY F. CAJORI

James Gregory is rightly credited with the introduction of the name "convergent" series. According to Reiff,[*] whom I have followed on this matter in my writings, Gregory also introduced the name "divergent" series. When a few years ago a correspondent raised the question whether Gregory really did use the word "divergent," I was not able to answer definitely because I could not secure access to a copy of Gregory's *Vera Circvli et Hyperbolae Quadratura,* in which the term occurs, according to Reiff. At last, I have found and examined a copy (Patavia, 1668), in the Naval Observatory at Washington, D. C.; the phrase "divergent series" does *not* occur in it, nor in Gregory's *Geometriae Pars Vniversalis* (Patavia, 1668), although in the *Vera Circvli, etc.,* "convergent" is used in the form of an adjective, verb, or noun over a hundred times. This term did not meet with immediate acceptance, for in 1705 [†] and again in 1713 [‡] Leibniz used the words "advergens," "advergentia" to signify convergent and convergence. These words did not cling to the mathematical phraseology. As regards the term "divergent," it was Nicolaus Bernoulli [§] who in 1713 first used "divergens" and "divergentia seriei," as is correctly stated in Cantor [||] and in the ENCYCLOPÉDIE.[¶]

THE UNIVERSITY OF CALIFORNIA.

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[*] R. Reiff, *Geschichte der Unendlichen Reihen,* Tübingen, 1889, p. 16.

[†] Leibniz's letter to J. Hermann, of April 7, 1705. See *Leibnizens Mathematische Schriften* (Ed. C. I. Gerhardt), vol. IV, p. 272. See also BIBLIOTHECA MATHEMATICA, (3), vol. 5 (1904), p. 308.

[‡] Leibniz's letter to Nicolaus Bernoulli, of June 28, 1713. See *Leibnizens Mathematische Schriften,* vol. III, p. 985.

[§] Nicolaus Bernoulli's letter to Leibniz, April 7, 1713. See *Leibnizens Mathematische Schriften,* vol. III, p. 983.

[||] M. Cantor, *Vorlesungen über Geschichte der Mathematik,* vol. III (2d ed.), p. 369.

[¶] ENCYCLOPÉDIE, vol. 1, 1907, p. 184, note 199.

A QUALITATIVE DEFINITION OF THE TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS*

BY P. FRANKLIN

1. *Introduction.* The object of the present note, which was suggested by a remark of Professor Birkhoff as to possible definitions of the trigonometric functions, is to define the trigonometric and hyperbolic functions by properties which shall be simple in the sense of not involving any of the ideas of the calculus, and qualitative in the sense of not involving relations as definitely explicit as functional or differential equations.†

2. *Postulates.* The characteristic properties (or postulates) which we use in our definition are stated in terms of a linear family of functions depending on two parameters, and they so restrict the family that each of its members is of the form $rF(mx + t)$ or the sum of two such expressions, where $F(x)$ is the function we wish to define. In proving this we shall incidentally give explicit rules for constructing the function $F(x)$ in terms of the family. We assume as the characteristic properties:

I. The two-fold linear family of functions $AG(x) + BH(x)$ (where $G(x)$ and $H(x)$ are any two independent members of the family) is independent of the choice of origin and direction of the x-axis; i.e., it is identical, as a family, with that given by $AG(x + c) + BH(x + c)$ or $AG(- x) + BH(- x)$.

II. Some pair of independent members of the family, as $G(x)$ and $H(x)$, are functions continuous for at least one value of x.

To these we shall add one of the following:

III (a). No member of the family never vanishes.

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* Presented to the Society, December 27, 1922.

† For definitions of the trigonometric functions involving such relations, see Osgood, *Lehrbuch der Funktionentheorie*, 1920, vol. 1, pp. 571–591.

III (*b*). There exists a member of the family which never vanishes, and all members of the family which never vanish are linearly dependent on this one, i.e., constant multiples of it.

III (*c*). There exist two linearly independent members of the family which never vanish.

We shall show that for I, II and III (*a*) the family is $A \sin mx + B \cos mx$, and hence each member is of the form $r \sin (mx + t)$. For I, II and III (*b*) the family is $A + Bx$, and hence each member of the family is of the form $mx + t$. Finally, for I, II and III (*c*) the family is $A \sinh mx + B \cosh mx$, and hence each member is of the form $r \sinh (mx + t)$; $r \cosh (mx + t)$ or $r \sinh (mx + t) \pm r \cosh (mx + t)$. Thus, after showing how to pick out the particular pair of functions desired from the family, we may define the trigonometric or hyperbolic functions by adjoining III (*a*) or III (*c*) respectively to I and II. It is interesting to note that as III (*a*), (*b*) and (*c*) are mutually exclusive, the conditions I and II alone define either the trigonometric functions, the hyperbolic functions, or the linear function, the last of which may be considered as a limiting case of either type. This result is natural, in view of the fact that I and the assumption that our functions admit two derivatives would practically restrict them to be solutions of a linear homogeneous differential equation of the second order with constant coefficients. It is interesting, however, to see that the much weaker restriction II is sufficient to give the result stated.

3. *Consequences of* I *and* II. Before proceeding to the separate cases mentioned above, we shall derive from I and II the fact that each member of the family is continuous at all points. To see this, let $G(x)$ and $H(x)$ be the functions mentioned in II, and let them be continuous at x_1. Then

$$G(x + x_2 - x_1) = A_1 G(x) + B_1 H(x),$$

since it is a member of the family by I, where x_2 is any number and A_1 and B_1 are constants depending on $x_2 - x_1$. But since the right member of this equation is continuous at $x = x_1$, the left member is continuous there, and $G(x)$ is continuous at x_2, any point. Similarly $H(x)$, and therefore every member of the family, is continuous everywhere.

4. *Consequences of* I, II, III (*a*). We now assume I, II and III (*a*). Let $F(x)$ be some member of the family (not identically zero) and let $F(b)$ be different from zero. We set

$$\frac{F(b+x)+F(b-x)}{2F(b)}=C(x). \tag{1}$$

It follows from this that $C(x) = C(-x)$; and that $C(0) = 1$. $C(x)$ vanishes by III (*a*), and as it is continuous, its zeros form a closed set. Let p be the smallest positive value of x for which $C(x)$ vanishes; then $C(p) = C(-p) = 0$ and we set:

$$C(x-p) = S(x). \tag{2}$$

From this we see that $S(0) = 0$; $S(p) = 1$; $S(2p) = 0$. Since $S(x)$ and $C(x)$ are not zero simultaneously, they are independent functions, and therefore every member of the family is a linear combination of them. But, by I, $S(x+h)$ and $C(x+h)$ are members of the family, and we may set

$$S(x+h) = AS(x) + BC(x), \tag{3}$$

$$C(x+h) = A'S(x) + B'C(x). \tag{4}$$

Putting in succession $x = 0$, $x = p$ in (3), and taking account of the particular values previously found, we obtain:

$$S(h) = B, \qquad S(h+p) = C(h) = A, \tag{5}$$

and (3) becomes:

$$S(x+h) = S(x)C(h) + S(h)C(x). \tag{6}$$

We may use this to obtain the value of $S(-p)$. For if we put $h = -x$, we obtain:

$$0 = S(x-x) = S(x)C(-x) + S(-x)C(x),$$
$$0 = C(x)[S(x) + S(-x)].$$

Consequently, $S(x) = -S(-x)$ for all values of x such that $C(x) \neq 0$, and hence, by the definition of p, for all values of x such that $-p < x < p$. But, since $S(x)$ is continuous at all points, we have also

$$S(-p) = -S(p) = -1.$$

This enables us to evaluate the coefficients in (4) by putting in succession $x = 0$, $x = -p$. This gives

$$C(h) = B', \qquad C(h-p) = S(h) = -A', \tag{7}$$

and (4) becomes

$$C(x+h) = C(x)C(h) - S(x)S(h). \tag{8}$$

If we put $h = -x$ in (8), we find (at least for values of x numerically less than p)

$$1 = C^2(x) + S^2(x). \tag{9}$$

Consider now a value of x between 0 and p. For such a value, we know from the continuity of the functions and the definition of p that $S(x/2)$ and $C(x/2)$ are both positive. Consequently, by combining (8) (for $h = x$) and (9), we may obtain the formulas:

$$S(x/2) = \sqrt{\frac{1 - C(x)}{2}}, \qquad C(x/2) = \sqrt{\frac{1 + C(x)}{2}}. \tag{10}$$

Combining these relations with (6) and (8), and the values obtained above, $S(0) = 0$, $S(p) = 1$, $C(0) = 1$, $C(p) = 0$, we may calculate by a finite number of operations the values of $S(x)$ and $C(x)$ for any value of x, which divided by p gives a proper fraction expressible as a terminating decimal in the binary scale. Again, since the relations and values used are all satisfied by $S(x) = \sin(\pi x/2p)$, $C(x) = \cos(\pi x/2p)$, the values so obtained will agree with the values for these functions. But, since $S(x)$ and $C(x)$ are continuous, and are equal to these functions for a set of values everywhere dense in the interval $(0, p)$, they must be identical with these functions throughout the entire interval.

Finally, by using the relations (6) and (8), and taking for h, p or $-p$ (recalling that $C(-p) = 0$, $S(-p) = -1$), we may show that if the functions given above represent $S(x)$ and $C(x)$ in any interval, they represent them in an interval longer by p in each direction, and hence for all values of x. This justifies our contention that under assumptions I, II and III (a) the family $AG(x) + BH(x)$ is of the form $A \sin mx + B \cos mx$. Furthermore, if the family is given, we may determine $S(x)$ and $C(x)$ as above, as well as p, and then define $\sin x$ and $\cos x$ by the relations: $\sin x = S(2px/\pi)$; $\cos x = C(2px/\pi)$.

5. *Consequences of* I, II, III (*b*). We next deduce the consequences of I, II, III (*b*). If $F(x)$ is the non-vanishing solution given by III (*b*), $F(x+c)$ and $F(-x)$, which are members of the family by I and evidently never vanish, must be linearly dependent on $F(x)$, giving:

$$F(x+c) = kF(x); \quad F(-x) = jF(x). \tag{11}$$

As the second equation reduces to $F(0) = jF(0)$ for $x = 0$, we see that $j = 1$. Combining this result with the first, we have, taking $c = (x_1 + x_2)/2$,

$$\begin{aligned} F(x_1) = F\left(\frac{x_1+x_2}{2} + \frac{x_1-x_2}{2}\right) &= kF\left(\frac{x_1-x_2}{2}\right) \\ &= kF\left(\frac{x_2-x_1}{2}\right) \\ = F\left(\frac{x_1+x_2}{2} + \frac{x_2-x_1}{2}\right) &= F(x_2). \end{aligned} \tag{12}$$

Since x_1 and x_2 are arbitrary, this shows that the non-vanishing solution $F(x)$ is a constant, and we may take the function $F(x)$ equal to 1.

If $E(x)$ is some non-constant member of the family, it must take on both positive and negative values, since otherwise by adding or subtracting a constant, we should obtain a second member of our family which never vanishes. Hence it must vanish at some point in such a way that the function is positive for all values of the variable in some left (or right) neighborhood of this point, and that in the right (or left) neighborhood of the same length this is not the case. Let b be the value of x at this point, and set

$$P(x) = E(b+x), \tag{13}$$

so that $P(0) = 0$; and note that all members of the family may be expressed in the form $A_1P(x) + B_1$. Then, in particular,

$$P(-x) = jP(x) + k; \tag{14}$$

and, on putting $x = 0$, we see that k is zero, and $P(-x) = jP(x)$. Since

$$P(x) = jP(-x) = j^2P(x), \tag{15}$$

we see that $j = +1$ or -1, and it cannot be $+1$, from the way in which b was selected, for this value used with (13) and (15) would give

$$E(b + h) = E(b - h),$$

and hence the behavior of $E(x)$ would be the same on the right and left neighborhoods of b. Hence $j = -1$, and

$$P(x) = -P(-x). \tag{16}$$

To obtain the addition theorem for $P(x)$, we notice from I that

$$P(x + h) = AP(x) + B. \tag{17}$$

On taking in succession $x = 0$, and $x = -h$, using (16) in the latter case, we obtain:

$$P(h) = B, \qquad 0 = P(0) = -AP(h) + B. \tag{18}$$

These relations show that $A = 1$, and reduce (17) to

$$P(x + h) = P(x) + P(h). \tag{19}$$

If we set $P(1) = m$, we see from (19) and (16) that, for rational values of x, the values of $P(x)$ equal those of mx. Hence, from the continuity of the function, we have $P(x) = mx$ for all values. Therefore, we have proved that the family is $A + Bx$ in this case.

6. *Consequences of* I, II, III (c). Finally let us consider the consequences of I, II and III (c). Let $F(x)$ be a non-vanishing member of the family, and let c be a number such that

$$F(x + c) \neq kF(-x + c). \tag{20}$$

There must be some choice of the function $F(x)$ and c, for if (20) were not true for some value of c, the reasoning used at the beginning of the discussion of the last case would show that $F(x)$ reduced to a constant, and if this held good for all the non-vanishing solutions, they would all be linearly dependent, and III (c) would be violated. We set

$$\begin{aligned} \frac{F(x + c) + F(-x + c)}{2F(c)} &= C(x), \\ \frac{F(x + c) - F(-x + c)}{K} &= S(x), \end{aligned} \tag{21}$$

where K is a constant to be determined later. It follows from the definitions that

$$C(0) = 1, \qquad S(0) = 0, \tag{22}$$
$$C(x) = C(-x), \qquad S(x) = -S(-x).$$

As before, we obtain the addition theorems by noting that

$$S(x + h) = AS(x) + BC(x), \tag{23}$$

$$C(x + h) = A'S(x) + B'C(x), \tag{24}$$

and evaluating the coefficients. On putting $x = 0$ in (23), we find, by using (22), that

$$S(h) = B; \tag{25}$$

while, on putting $x = -h$, we find

$$0 = S(h - h) = -AS(h) + BC(h). \tag{26}$$

The last two equations show that

$$0 = S(h)[C(h) - A], \tag{27}$$

and hence that $A = C(h)$ when $S(h) \neq 0$. Thus, with this restriction, (23) becomes

$$S(x + h) = S(x)C(h) + S(h)C(x). \tag{28}$$

We remove the restriction by noting that if $S(h) = 0$, while $S(x)$ is not 0, we may interchange the roles of x and h; finally, if $S(x) = S(h) = 0$, (23) and (25), which hold in all cases, show that $S(x + h) = 0$, and thus (28) holds good in this case.

If we put $x = 0$ in (24), we find $C(h) = B'$, and (24) becomes

$$C(x + h) = A'S(x) + C(x)C(h). \tag{29}$$

If we observe that A' is a function of h, interchange x and h and subtract, we find

$$A'(h)S(x) = A'(x)S(h). \tag{30}$$

This is an identity in x and h. If we select a value of x, x_1 for which $S(x_1) \neq 0$, and put $A'(x_1)/S(x_1) = k$, we see that $A'(h) = kS(h)$, and (29) becomes

$$C(x + h) = C(x)C(h) + kS(x)S(h). \tag{31}$$

The number k is positive, negative, or zero. The last case cannot occur, since then we should have

$$C(x + h) = C(x)C(h),\ 1 = C(h - h) = C^2(h),\ C(h) = 1. \tag{32}$$

But this would show that $C(h)$ was a constant, and would reduce (28) to (19), which would prove the family to be

$A + Bx$. This would not satisfy III (c). Also k cannot be negative ($= -k'$), for then we should have

$$C(x+h) = C(x)C(h) - k'S(x)S(h), \quad 1 = C(h-h) = C^2(h) + k'S^2(h). \tag{33}$$

The second equation shows that for a value of h for which $S(h) \neq 0$, $C(h)$ is less than 1. But $C(h)$ is always positive, since it never vanishes. Since it is continuous, it takes on all values between this value and unity. Since $\cos(\pi/2^n)$ is nearer unity than any fixed number for some value of n, we may find an n and an h_1 such that

$$C(h_1) = \cos(\pi/2^n), \tag{34}$$

and since, as a consequence of (33), we have

$$C(2h) = 2C^2(h) - 1, \tag{35}$$

which is a relation satisfied by the cosine, we see that

$$C(2^{n-1}h_1) = \cos(\pi/2) = 0, \tag{36}$$

which shows that $C(h)$ is not a non-vanishing solution, and hence that the assumption that k is negative leads to a contradiction.

Since k is positive, and becomes k/m^2 if the factor K in the definition of $S(x)$ is replaced by K/m, it may be made unity by a suitable choice of K, and (31) becomes

$$C(x+h) = C(x)C(h) + S(x)S(h). \tag{37}$$

For $h = -x$, this becomes

$$1 = C(x-x) = C^2(x) - S^2(x), \tag{38}$$

while for $h = x$, we get, using (38),

$$C(2x) = 2C^2(x) - 1, \qquad C(x/2) = \sqrt{\frac{1+C(x)}{2}}, \tag{39}$$

where the sign is determined by the fact that $C(x)$ is always positive. Let $C(x_1)$ be a value of $C(x)$ different from unity, and hence greater than unity by (38). Then $C(x_1) = \cosh mx_1$, for some positive m, since $\cosh x$ takes on all values greater than 1 for positive values of x. Hence we see from

$$C(0) = \cosh 0, \qquad C(x_1) = \cosh mx_1, \tag{40}$$

and from the fact that (39) is true for the function $\cosh mx$, that this function has the same values as $C(x)$ for all values of the form $2^n x_1$ or $x_1/2^n$. Also for these values, we see from (38) that $S(x) = j \sinh mx$, where j is plus or minus one, and must be the same for all such values of x, since (28) shows that

$$S(2x) = 2S(x)C(x). \tag{41}$$

Finally, since (28) and (37) are the addition formulas for $\sinh mx$ and $\cosh mx$, we see that these functions agree with $S(x)$ (to within a sign) and $C(x)$ for all multiples of x_1 whose fractional parts are terminating decimals in the binary scale, and hence, since all the functions concerned are continuous, at all points. Thus under I, II and III (c), the family is necessarily $A \sinh mx + B \cosh mx$.

7. *Conclusions.* In conclusion, we notice that since I and II alone must determine one of the three types of families discussed, we may use any characteristic property of the types in place of III. Thus, we might replace III (a) by the assumption "Some member of the family vanishes twice," or "Every member of the family is bounded." This last statement may be extended so as to give an alternative form of the assumption III, in terms of bounded, instead of non-vanishing functions. That is, III (a), (b) and (c) above may be replaced by the following postulates:

III (a'). There exist two linearly independent members of the family which are bounded.

III (b'). There exists one member of the family which is bounded, and all other bounded members of the family are linearly dependent on this one.

III (c'). No member of the family is bounded.

HARVARD UNIVERSITY

GROUPS IN WHICH THE NUMBER OF OPERATORS IN A SET OF CONJUGATES IS EQUAL TO THE ORDER OF THE COMMUTATOR SUBGROUP*

BY G. A. MILLER

1. *Introduction.* From the fact that the commutator quotient group is abelian, it results directly that there is no

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* Presented to the Society, Sept. 7, 1922.

group in which the number of operators in a complete set of conjugates is larger than the order of the commutator subgroup. Hence the groups which satisfy the condition noted in the heading of this paper contain the largest possible number of operators in a complete set of conjugates. They evidently include all the abelian groups as a special case, since the commutator subgroup of every abelian group is the identity. On the other hand, they do not include any simple group of composite order, since such a group is identical with its commutator subgroup.

Each of the groups which satisfy the condition noted in the heading of this paper must involve a co-set as regards its commutator subgroup such that all the operators of this co-set have the same order. Moreover, whenever the commutator subgroup contains more than one Sylow subgroup of a given order, it must transform these Sylow subgroups according to a simply transitive group. If this were not the case, all the operators in a co-set with respect to the commutator subgroup could not transform these Sylow subgroups according to substitutions of the same degree. This common degree would clearly have to be equal to the number of the Sylow subgroups of the same order diminished by one for every set of conjugate Sylow subgroups found in the commutator subgroup.

From the preceding paragraph it results that if a group G contains a set of conjugate operators whose number is equal to the order of the commutator subgroup of G, then this subgroup must satisfy certain conditions. In fact, some groups cannot be the commutator subgroup of any group whatsoever. For instance, it is easy to prove that no dihedral group whose order exceeds 4 can be a commutator subgroup. In fact, none of the non-invariant operators of order 2 contained in such a group could be a commutator since the product of such an operator and some other operator could not transform the generators of the characteristic cyclic subgroup into the same power of themselves as the latter operator. It is also easy to prove that no symmetric group whose order exceeds 2 can be the commutator subgroup of a group.

In fact, if a symmetric group were the commutator subgroup of a group, an operator t outside of this symmetric group would have to be transformed into itself multiplied by an operator represented by a negative substitution of the symmetric group. It is evident that this product could not correspond to t in an automorphism of the group when the order of the symmetric group is neither 2 nor 720. In the special case when the order of the symmetric group is 720 this group admits 720 outer isomorphisms. If t transformed the symmetric group according to an outer isomorphism it results again that the given product could not correspond to t in an automorphism since this product would transform the operators of order 5 in the symmetric group into a different power than t does. While no symmetric group whose order exceeds 2 can be the commutator subgroup of any group whatever, every alternating group is the commutator subgroup of the corresponding symmetric group. An alternating group whose order exceeds 3 cannot, however, be the commutator subgroup of a group in which the number of operators in a set of conjugates is equal to the order of the commutator subgroup. It is interesting to inquire what groups can be commutator subgroups of such groups. In the next section we shall prove that every abelian group has this property.

2. *Abelian Commutator Subgroups.* Let H be an arbitrary abelian group of order h. It is not difficult to construct a group G which has H for its commutator subgroup and involves a set of h conjugate operators. When h is odd such a G can evidently be found by adjoining to H an operator of order 2 which transforms every operator of H into its inverse. The group thus obtained is clearly the generalized dihedral group of order $2h$.

The generalized dihedral group obtained by extending H by means of an operator of order 2 which transforms every operator of H into its inverse has evidently always for its commutator subgroup the group generated by the squares of the operators found in H, and the number of operators in each of its complete sets of conjugate non-invariant operators of order 2 is equal to the order of this commutator subgroup. When h is even we can therefore construct the required G by first extending H by operators which are commutative with every operator of H and have for their squares the various operators in a set of independent generators of H whose orders are of the form 2^α. By extending the abelian group H thus obtained by means of an operator of order 2 which transforms each of its operators into its inverse we obtain a group which has H for its commutator subgroup and in which the number of operators in the complete sets of conjugate non-invariant operators of order 2 is equal to h.

The order of the G thus constructed is equal to the order of H multiplied by $2^{\beta+1}$, where β is equal to the number of the invariants of H which are of the form 2^α. All the operators whose orders exceed 2 in this G are conjugate in pairs, while the number of the complete sets of h conjugates is 2^β. It is easy to construct by other methods groups which satisfy the given conditions. For instance, if we extend the generalized dihedral group of order $2h$, obtained by extending H by means of an operator of order 2 which transforms every operator of H into its inverse, by means of the operators in its group of isomorphisms which are commutative with every operator of H, there results a group of order $2h^2$ which satisfies the conditions in question.

3. *Number of Conjugates Equal to the Order of the Commutator Subgroup Diminished by One.* The general problem of determining all the groups which satisfy the conditions expressed in the heading of the present paper seems to be quite difficult. There are, however, various special cases which may possibly contribute towards this general solution and which lead to interesting results. Among these is the case when the group G contains a set of conjugate operators whose number is one less than the order k of its commutator subgroup K, in addition to containing a set whose number is equal to k. It is easy to see that the former condition implies the latter while the converse is evidently not true.

In fact, if G involves a set of $k-1$ conjugates, the remaining operator in the co-set with respect to K to which these conjugates belong must be an invariant operator. Hence K must involve $k - 1$ conjugates under G and it must therefore be an abelian prime power group of type $(1, 1, 1, \cdots)$. Since the operators of K are transformed according to a transitive substitution group of degree $k - 1$ under G, there must be an operator of G which transforms them according to a substitution of degree $k - 1$. As this operator is commutative with only the identity in K it has k conjugates under G. Hence the following theorem has been established:

THEOREM. *If a group contains a set of conjugate operators whose number is equal to the order of the commutator subgroup diminished by one, it must also contain a set of conjugate operators whose number is equal to the order of its commutator subgroup, and this subgroup must be an abelian prime power group of type* $(1, 1, 1, \cdots)$.

No co-set of G with respect to K involves more than one invariant operator, and when it involves one such its remaining operators constitute a single set of conjugates under G. The invariant operators in all of these co-sets constitute the central of G and this has only the identity in common with K. Hence G involves the abelian group which is the direct product of K and the central of G. Each of the operators of G which does not appear in this direct product has k conjugates under G, and the order of G is equal to k times the order of this direct product since the $k - 1$ operators of prime order contained in K are transformed under G according to a regular group. The fact that this group is regular results directly from the facts that it is transitive and abelian.

Whenever a group contains a commutator subgroup of prime order p it must also contain at least one set of p conjugates, but when the commutator subgroup is neither of prime order nor the identity the number of operators in a set of conjugates is not necessarily equal to the order of the commutator subgroup. If a group whose commutator subgroup is of order p contains also sets of conjugates involving more than one but less than p operators, each of its co-sets with respect to K which involves less than p conjugates involves just one invariant operator. The other operators of such a co-set are conjugate in sets of k, where k is a divisor of $p-1$ and has the same value for all such sets.

The fact that the commutator subgroup of G must be abelian whenever G contains a set of conjugate operators whose number is one less than the order of its commutator subgroup may also be regarded as a special case of the theorem that in any group whatever, the difference between the order of the commutator subgroup and the number of operators in a complete set of conjugates is a multiple of the number of the conjugates of any one of the operators of this set under the commutator subgroup. In particular, when there is a set of $k-1$ conjugate operators they must be commutative with every operator of K and hence K must be abelian.

4. *The Number of Conjugates of Every Non-Invariant Operator is Equal to the Order of the Commutator Subgroup.* When the number of operators in every complete set of non-invariant conjugates of the group G is equal to k, the order of its commutator subgroup K, it follows directly that K must be in the central of G since none of its operators could have k conjugates under G. To prove that K must be a prime power group it is only necessary to note that if s is any non-invariant operator of G then the operators of G which are commutative with s constitute an invariant subgroup of index k under G, and the corresponding quotient group is simply isomorphic with K since s transforms the operators of G into themselves multiplied by the various operators of K.

If the order of K is not of the form p^m, p being a prime number, let q and p be two different prime divisors of k. To an operator of order q in the given quotient group there must correspond an operator of order q^β in G. As this operator would have k conjugates under G it would have to be transformed under G into itself multiplied by an operator of order p contained in the central of G. This is evidently impossible since the order of this product could not be of the form q^β Hence we have established the following theorem:

THEOREM. *If the number of operators in every set of non-invariant conjugates of a group is equal to the order of the commutator subgroup, then the order of this subgroup is of the form* p^m, *p being a prime number.*

From the preceding theorem it results directly that G is the direct product of an abelian group and a non-abelian group whose order is of the form p^m. It is easy to prove that K must be of type $(1, 1, 1, \cdots)$. If this were not the case the operator s of G which was defined above would transform an operator t_1 into itself multiplied by an operator of order p^α while it would transform t_2 into itself multiplied by an operator of order $p^{\alpha+1}$ whose pth power is the multiplier of t_1 under s, $p^{\alpha+1}$ being an operator of highest order contained in K. Hence it may be assumed that $t_2^p = t_1$. This is impossible since G would also have to contain an operator which would transform t_1 into itself multiplied by an operator of order $p^{\alpha+1}$. This proves also incidentally that the pth power of every operator of G is found in the central of G.

In the special case when $k = p$ it is easy to prove that the order of the central quotient group of the Sylow group P of order p^m would have for its order an even power of p. In fact, all the operators which are commutative with a non-invariant operator s_1 of P constitute a subgroup of index p. If s_2 is any operator in P which is not commutative with s_1, then the cross-cut of the subgroups composed of all the operators which are commutative with s_1 and s_2 respectively will be of index p^2 under P. This cross-cut includes the central of P and if it includes other operators we may find a subgroup of index p^2 contained in it and involving the central of P. As this process may be continued until we arrive at the central the theorem in question has been proved.*

THE UNIVERSITY OF ILLINOIS

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* Cf. J. W. A. Young, AMERICAN JOURNAL, vol. 15 (1893), p. 171.

ON CURVES KINEMATICALLY RELATED TO A GIVEN CURVE*

BY H. PORITSKY

In a space S is given a curve C. With any point P on it is associated a position of the moving trihedron formed by the tangent, principal normal, and binormal to the curve at P. We might call an *indicatrix trihedron* a trihedron whose axes pass through a fixed point, and are parallel to those of the moving trihedron, and denote it by I. As the point P describes the curve, the indicatrix trihedron will rotate about (lines passing through) its vertex. This motion, or the motion of a space rigidly connected to the indicatrix trihedron, with reference to the space S, we shall denote by $[I, S]$. The problem solved in this paper is, to find all the curves C' such that their indicatrix trihedra I' (when drawn with the vertex as I) will experience a motion $[I', S]$ identical with the motion $[I, S]$; in other words, to find curves C', whose points P' can be made to correspond to P so that, as the curves are described by corresponding points, their indicatrix trihedra I, I' remain relatively invariant. Certain interesting families of curves C' are shown to exist. A kinematic method of treatment has been adopted.

Denote unit vectors along the tangent, principal normal, and binormal of C by $\mathbf{i}, \mathbf{j}, \mathbf{k}$,† respectively. It is known that the motion $[I, S]$ is completely characterized by the fact that there is no component of rotation along the principal normal $\mathbf{j}$.‡ The components of the rotation along $\mathbf{i}$, $\mathbf{k}$, if P describes C with unit velocity, may be identified with the torsion τ and the curvature κ. If the velocity of P is not unity but $v = ds/dt$ (s is arc length, t time), the components of rotations become $v\tau$, $v\kappa$. Thus the rotation vector always lies in the plane π, through $\mathbf{i}$, $\mathbf{k}$, and parallel to the rectifying plane. It will be noticed that π is a plane of (i.e., fixed relative to) the moving space I'.

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* Presented to the Society, December 28, 1922.

† Clarendon type will be used to denote vectors.

‡ Darboux, *Théorie des Surfaces*, 2d ed., vol. I, p. 13.

We must now consider two cases:

1. The motion $[I, S]$ consists of a rotation about a fixed line.

2. The motion $[I, S]$ may be generated by rolling a cone fixed in I upon a cone fixed in S. The cones are the loci of instantaneous axes of revolution in I and S respectively. Case 1 is the special instance when the cones degenerate into lines.

Since, as just mentioned, the rotation vector has no component along $\boldsymbol{j}$, the rolling cone, whose elements bear the rotation vectors, reduces to the plane π (case 2), or else the fixed axis of rotation is a line in π (case 1). In the latter case the components of rotation along $\boldsymbol{i}$, $\boldsymbol{k}$ have a fixed ratio. Hence

$$\frac{\kappa}{\tau} = \text{const.} = \tan \alpha,$$

where α is the angle which the fixed rotation axis makes with $\boldsymbol{i}$. C is known to be a cylindrical helix. Leaving this case for later consideration, we shall first consider the general case.

Let primed letters indicate objects referring to C' of the same kind as the unprimed letters denote for C. Let corresponding points on the two curves be given by the same value of t; thus $P(t)$, $P'(t)$ will denote corresponding points on C, C', while P, P' denote merely any two points on C, C'; similarly $\boldsymbol{i}(t)$, $\boldsymbol{i}'(t)$ will be used for unit tangents at two corresponding points, and so forth.

If the motions $[I, S]$, $[I', S]$ be identical, the fixed and the rolling cones for the two motions, as well as the angular velocities, must be the same. But since the rolling cones reduce to the planes π, π' which are parallel to the rectifying planes at $P(t)$, $P'(t)$, respectively, $\pi(t)$, $\pi'(t)$ must coincide, and so must the principal normals $\boldsymbol{j}(t)$, $\boldsymbol{j}'(t)$ which are perpendicular to π, π'. The rotation by means of which $I(t)$ is transformed into $I'(t)$ ($\boldsymbol{i}(t), \boldsymbol{j}(t), \boldsymbol{k}(t)$ going into $\boldsymbol{i}'(t), \boldsymbol{j}'(t), \boldsymbol{k}'(t)$) is therefore about the line bearing $\boldsymbol{j}(t)$, $\boldsymbol{j}'(t)$, and is of a constant angle θ, since the two indicatrix trihedra are relatively fixed. Resolving the unit vectors of I' along those of I, we get

$$\begin{aligned} \boldsymbol{i}'(t) &= \boldsymbol{i}(t)\cos\theta + \boldsymbol{k}(t)\sin\theta, \\ \boldsymbol{j}'(t) &= \boldsymbol{j}(t), \\ \boldsymbol{k}'(t) &= -\,\boldsymbol{i}(t)\sin\theta + \boldsymbol{k}(t)\cos\theta, \end{aligned} \tag{1}$$

θ being independent of t.

Equating the rotation vectors

$$v'(t)[\tau'(t)\boldsymbol{i}'(t) + \kappa'(t)\boldsymbol{k}'(t)] = v(t)[\tau(t)\boldsymbol{i}(t) + \kappa(t)\boldsymbol{k}(t)], \tag{2}$$

we have identified the fixed cones and the angular velocities. Solving (1) for $\boldsymbol{i}$, $\boldsymbol{k}$, in terms of $\boldsymbol{i}'$, $\boldsymbol{k}'$, substituting in (2) and equating coefficients of like vectors we get

$$\begin{aligned} v'(t)\tau'(t) &= v(t)\tau(t)\cos\theta + v(t)\kappa(t)\sin\theta, \\ v'(t)\kappa'(t) &= -\,v(t)\tau(t)\sin\theta + v(t)\kappa(t)\cos\theta. \end{aligned} \tag{3}$$

Nothing is lost in generality if we identify the arc length along the curve C with time, thus letting $v(t) = 1$.

Having chosen θ, the successive positions of I' and consequently also of $\boldsymbol{i}'$ are determined as functions of t. The curve C' itself is not specified till we give $v' = ds/dt$, whereupon we obtain C' by vectorial integration of $v'(t)\boldsymbol{i}'(t)$ (one must also give the starting point). All the curves that we obtain from a fixed θ by varying v' can be made to correspond to each other, so that at corresponding points (given by the same values of t), the indicatrix trihedra coincide, and are thus to each other in relation of Combescure.

We shall restrict ourselves for the present to the case where $v' = 1$, but will let θ take on any value between 0 and 2π; the curves C', which we are seeking, will have any two of their points the same arc length apart as the points of C to which they correspond. We may now put $s' = s = t$ where s' is the arc length along C; (3) reduces to

$$\begin{aligned} \tau'(t) &= \tau(t)\cos\theta + \kappa(t)\sin\theta, \\ \kappa'(t) &= -\,\tau(t)\sin\theta + \kappa(t)\cos\theta. \end{aligned} \tag{3'}$$

If we start all the curves C' with the point $s' = 0$ at a common starting point Q, there will be obtained a one-parameter family of curves F, one curve corresponding to every value of the parameter θ. The given curve C corresponds to $\theta = 0$. All other curves that have an indicatrix trihedron I' fixed relative to I may be obtained from some one curve of F by a transformation of Combescure. (3′) shows that the curvatures and torsions of the curves of F are certain linear combinations of those of the initial curve. The family F has some interesting properties which we now proceed to investigate.

Denote by $\boldsymbol{P}(\theta, s)$ the vector drawn from the point Q to any point P' on the curves F; similarly replace $\boldsymbol{i}'(t)$, $\boldsymbol{j}'(t)$, $\boldsymbol{k}'(t)$ by $\boldsymbol{i}(\theta, s)$, $\boldsymbol{j}(\theta, s)$, $\boldsymbol{k}(\theta, s)$. Between $\boldsymbol{P}$ and $\boldsymbol{i}$ we have the relation

$$\frac{\partial \boldsymbol{P}(\theta, s)}{\partial s} = \boldsymbol{i}(\theta, s), \tag{4}$$

whence by integration, since $\boldsymbol{P}(\theta, 0) = 0$,

$$\boldsymbol{P}(\theta, s) = \int_0^s \boldsymbol{i}(\theta, s) ds. \tag{4'}$$

Using (1) we obtain

$$\boldsymbol{P}(\theta, s) = \cos \theta \int_0^s \boldsymbol{i}(0, s) ds + \sin \theta \int_0^s \boldsymbol{k}(0, s) ds. \tag{5}$$

Since $\boldsymbol{k}(\theta, s) = \boldsymbol{i}(\theta + \frac{\pi}{2}, s)$, (5) becomes

$$\boldsymbol{P}(\theta, s) = \cos \theta \int_0^s \boldsymbol{i}(0, s) ds + \sin \theta \int_0^s \boldsymbol{i}\left(\frac{\pi}{2}, s\right) ds, \tag{5'}$$

and, using (4'), we find

$$\boldsymbol{P}(\theta, s) = \cos \theta \, \boldsymbol{P}(0, s) + \sin \theta \, \boldsymbol{P}\left(\frac{\pi}{2}, s\right). \tag{6}$$

If we hold s constant and let θ run between 0 and 2π, the end point of $\boldsymbol{P}(\theta, s)$ will describe the locus of the corresponding points on the curves F, which are at an arc length s from Q. From (6) we conclude that these corresponding points always lie on ellipses E. The latter have their centers at Q, and the vectors $\boldsymbol{P}(0, s)$, $\boldsymbol{P}(\pi/2, s)$ are conjugate semi-axes. Obviously, any two curves whose values of θ differ by $\pi/2$ will yield conjugate axes of the ellipses. We might call such curves conjugates of each other. The tangents to two conjugate curves are perpendicular to each other.

When s is small the ellipses approximate a circle. For, from (5), $\boldsymbol{P}(\theta, s) = [\cos \theta \; \boldsymbol{i}(0, 0) + \sin \theta \; \boldsymbol{k}(0, 0)]s$ approximately for small s. The plane of this circle is perpendicular to the common principal normal of F at Q.

In the plane $\pi(0)$ through $\boldsymbol{i}(0, 0)$, $\boldsymbol{k}(0, 0)$, let us set up a system of rectangular Cartesian axes (x, y) with origin at the vertex of I, the end-points of the two unit vectors mentioned having coordinates $(1, 0)$, $(0, 1)$, respectively. The unit circle described by $\boldsymbol{i}(\theta, 0)$ now has for its parametric equations

$$x = \cos\theta, \qquad y = \sin\theta. \tag{7}$$

Similarly, in the plane of the ellipse E, described by $\boldsymbol{P}(\theta, s)$ when s is held fixed, let us set up a system of oblique Cartesian coordinates (ξ, η) the origin being at Q, while the axes and scales on them are chosen so that the coordinates of $\boldsymbol{P}(0, s)$, $\boldsymbol{P}(\pi/2, s)$ are $(1, 0)$ and $(0, 1)$, respectively. Any vector $a\boldsymbol{P}(0, s) + b\boldsymbol{P}(\pi/2, s)$ will then have for the coordinates of its end-points $\xi = a$, $\eta = b$. Hence, using (6) we get for the parametric equations of the ellipse,

$$\xi = \cos\theta, \qquad \eta = \sin\theta. \tag{8}$$

Now consider any affine projective transformation carrying the plane (x, y) into (ξ, η) so that for corresponding points $x = \xi$ and $y = \eta$. Such an affine projective transformation will send the circle (7) into the ellipse (8), corresponding points being given by the same value of θ. Therefore, corresponding points on two ellipses are projectively related. If we join such points by drawing corresponding secants for the curves, that is, secants between two corresponding points, we obtain a ruled surface of (not higher than) the fourth degree. By passing to the limit we get a similar result for the tangents.

The result obtained that points of F lie on ellipses E is only a special case of a more general theorem. If we call M the motion described by the moving trihedron of F as θ is varied but s is held fixed, and denote by R any point fixed to that trihedron, then the path of R as a result of the motion M will be an ellipse. For

$$\begin{aligned}\overrightarrow{QR} &= \overrightarrow{QP} + \overrightarrow{PR}\\ &= \boldsymbol{P}(\theta, s) + c_1\boldsymbol{i}(\theta, s) + c_2\boldsymbol{j}(\theta, s) + c_3\boldsymbol{k}(\theta, s),\end{aligned}$$

where c_1, c_2, c_3 are constants. Using (1) and (6) we obtain

$$\begin{aligned}\overrightarrow{QR} = {} & \cos\theta[\boldsymbol{P}(0, s) + c_1\boldsymbol{i}(0, s) + c_3\boldsymbol{k}(0, s)]\\ & + \sin\theta\left[\boldsymbol{P}\left(\frac{\pi}{2}, s\right) + c_1\boldsymbol{i}\left(\frac{\pi}{2}, s\right) + c_3\boldsymbol{k}\left(\frac{\pi}{2}, s\right)\right] + c_2\boldsymbol{j}(0, s);\end{aligned}$$

s being held fixed $= s_0$, this is the equation of an ellipse having its center at $c_2\boldsymbol{j}(0, s_0)$, that is, on the line through Q having the direction of the principal normals at the points $\boldsymbol{P}(\theta, s_0)$.

The movements M in which all points describe ellipses are of the type described by Darboux in one of his notes to Koenigs' *Kinematics*.* It is there proved that this is the only possible movement in which the trajectories of all points are plane curves, except for the case where the trajectories lie in a family of parallel planes. A line generally describes a surface of the fourth degree. Such a motion may be generated by proper rolling and slipping of two circular cylinders whose radii are in the ratio 1 : 2. The axes of these cylinders will in the present instance be parallel to the principal normals.

Since the relation (3′) between κ, τ and κ', τ' is the same as between the coordinates of the same point in a plane referred to two sets of rectangular axes with a common origin and forming an angle θ with each other, it follows that if κ, τ satisfy an algebraic relation, κ', τ' will satisfy a relation of the same degree. Thus, if κ, τ satisfy a linear relation so that C is a Bertrand curve, all the curves of F will be Bertrand curves. There will then exist two values of θ for which $\kappa' =$ const., and two other values differing from the former by $\pi/2$ for which $\tau' =$ const. The latter curves are the conjugates of the former.

The expression $[\kappa^2 + \tau^2]^{1/2}$ gives the rate at which the principal normal or the rectifying plane turns, and is sometimes spoken of as the "normal" curvature. From (3) it follows that for all the curves of F, the normal curvature is the same. In particular, if it is constant for C, it is equal to the same constant for any curve of F.

F was obtained by putting $v' = ds'/ds$ equal to unity, that is, by letting corresponding points on C and C' be equal arc lengths apart. Other interesting families of curves may, no doubt, be obtained by choosing other proper expressions for v'. Thus, if we let v' be a function of θ only, $v' = v(\theta)$, we can go through a treatment similar to that for the F family. Since we may no longer use the same symbol s to denote arc length along any curve, we shall think of corresponding points along different curves as given by the same value of t. We now have (in place of (4))

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* G. Koenigs, *Leçons de Cinématique*, Note IV, §§ 2, 3, p. 252.

$$\frac{\partial \boldsymbol{P}(\theta, t)}{\partial t} = \frac{\partial \boldsymbol{P}(\theta, t)}{\partial s'}\frac{ds'}{dt} = v(\theta)\cdot\frac{\partial \boldsymbol{P}}{\partial s'} = v(\theta)\boldsymbol{i}(\theta, t);$$

and, instead of (6),

$$(9) \qquad \boldsymbol{P}(\theta, t) = \cos\theta\, v(\theta)\boldsymbol{P}(0, t) + \sin\theta\, v(\theta)\boldsymbol{P}\left(\frac{\pi}{2}, t\right).$$

This is the locus of corresponding points on the curves now. It is still a plane curve and may be regarded as an affine projective image of the curve whose equation in polar coordinates is $\rho = v(\theta)$, this curve taking the place of the unit circle of which the ellipses of family F were projective images.

In case $\rho = v(\theta)$ is an algebraic curve, it can be shown that the movements M, described by the moving trihedra of the curves as θ varies while t is fixed, are still algebraic, that is, the path of any point during this motion is an algebraic curve; its degree is considerably higher than that of $\rho = v(\theta)$. Thus if we take $\rho = \sec\theta$, the corresponding points of the curves always lie on straight lines. During the motion M the general point can be shown to describe a curve of the fourth degree which is an affine projective image of

$$z = \tan\theta, \qquad x = \cos\theta, \qquad y = \sin\theta.$$

Now let us return to the case where C is a cylindrical helix, the motion $[I, S]$ reducing to a rotation of I about a fixed axis l (whose direction gives the direction of the rulings of the cylinder). The positions I' could take up relative to I are now of two parameters instead of one. For, if the position of $\boldsymbol{i}$ be arbitrarily assigned, since the rotation vector lying on l must always be in the plane $\boldsymbol{i}'$, $\boldsymbol{k}'$, the position of I' is completely specified (except for the case when $\boldsymbol{i}$ coincides with l). $I'(t)$ is now determined in time, and it remains only to choose $v'(t)$ and the initial point in order to specify C'. Whichever $I'(t)$ we pick out, it rotates about l, $\boldsymbol{i}$ forming a constant angle with it. Hence all the curves C' are cylindrical helices, the elements of whose cylinders are parallel to those of C. The curves obtained by putting $v' = 1$, and prescribing a common starting point Q form a two-parameter family of curves G, such that any other curve whose indicatrix I' is fixed relative to I is in a relation of Combescure to some curve of G.

If we restrict $\boldsymbol{i}$ to lie in a plane through l, we pick out a one-parameter group of positions of I, which is of the type previously discussed for a general curve C. We may now construct the family F by picking out a position of $\boldsymbol{i}$ to measure θ from. For two values of θ (differing by π) the unit tangent will coincide with l, yielding for the curve C a straight line C_s. The two curves C_p that are conjugate to C_s will have their tangents perpendicular to the line C_s, and consequently lie in a plane perpendicular to C_s. The vectors joining Q to corresponding points on C_s, C_p will form conjugate axes of the ellipse E, and being perpendicular to each other will now be the major and minor axes of E. The points of F can thus be obtained by having an ellipse move so that its major axis increases at a uniform rate along a straight line while the minor axis describes a plane curve with the same velocity.

If we now rotate about l the plane in which we restricted $\boldsymbol{i}$ to lie, all possible positions of I' (relative to I) are obtained. The infinitesimal vectors $\boldsymbol{i}ds$ experience a fixed rotation about l, and the same is true of the sum vector $\int_0^s \boldsymbol{i}ds$. Hence, all the other curves of G may be obtained from the one-parameter family just considered by rotating the latter about l.

We see, therefore, that if we hold s constant, the locus of a point fixed relative to I' is a surface of the fourth degree obtained by revolving an ellipse about the line l. In particular the locus of the vertex of I' which describes the curves C themselves reduces to an ellipsoid of revolution since the ellipse has its major axis on l. The torsions and curvatures of C' are again linear combinations of those of C, the relation being given by (3′). Hence, if C is a circular helix, so that τ, κ are constant, τ', κ' will also be constant, and all the curves of G will be circular helices.

Cornell University

HEATH ON GREEK MATHEMATICS

A History of Greek Mathematics. By Sir Thomas Heath. Volume I, *From Thales to Euclid;* volume II, *From Aristarchus to Diophantus*. Oxford, Clarendon Press, 1921. xvi + 446 + xii + 586 pp.

It is doubtful if anyone could assume the privilege of writing a review (at least an American review) of this monumental treatise without a feeling of helplessness. An American review must, by our present standard of taste in matters literary and critical, be brief, be popular, and (according to certain other standards) be both flippant and faultfinding; but no brief review can do this work justice nor can a sufficiently extended review be popular, while either flippancy or faultfinding would be merely a display of poor taste, like loud boasting or any other of the various species of vulgarity.

With such thoughts, a reviewer may properly ask himself what, precisely, is his mission. Is it to seek out points of doubt in the narrative and magnify them into blunders, or to search out minor inaccuracies (for such can always be found in any book) and display them as typical of the work under inspection? Should we take seriously Montaigne's *mot*, "Since we cannot attain to greatness, let us have our revenge by railing at it"? Or has the critic a more wholesome duty to his readers—that of stating the salient features of an author's work and of discovering whether he has seen things as they really are, has recorded them with becoming felicity of style, and has searched out such of the causes for world progress as can be shown to exist?

In considering the work under review, therefore, it is proposed to ask whether the author has seen Greek mathematics as it was, and whether he has expressed the results of his studies in a style that will command the willing attention of those whose tastes will lead them to read a work of this nature. In the seeing of Greek mathematics as it was, there is involved the question of causes, of the general setting of mathematics in the philosophical schools of that remarkable race, and of the balancing of the merits of different investigators and expositors. In the matter of style, a subject admitting of only slight mention, there may well be raised the question of the influence upon the author of the Greek language in which the science was expressed—a subject not without value in these days in which Greek is reported to have become obsolete as a subject of study in the schools of our country.

First, then, has the author seen the Greek mathematicians as it were face to face, intellect to intellect, as one master of the subject to another? —or has he seen them as in a glass, darkly, trusting to secondary sources and to unexplored tradition? The answer is a simple one—that no man now living is more capable than he of interpreting the Greek mathematical mind to the scholar of today; indeed, there is no one who ranks even in the same class with Sir Thomas Heath in this particular. Paul Tannery might have done so in his lifetime, but in knowing the mathematics of Greece, in knowing the Greek of mathematics, and in knowing the causes which led to the great development of the science in Miletus, in Crotona, in Athens, in Alexandria, and in the islands of the Ægean Sea, the author of this treatise stands without a rival, and these causes he has set forth for us in his usually lucid style.

This is a strong assertion, it sounds exaggerated, and it has the odor of that type of flattery from which Byron and many another of lesser fame has begged to be delivered—and yet the assertion is perfectly true. Sir Thomas Heath ranked in his university as a leader in mathematics and as a leader in Greek, and thus he was equipped as few others have been to see Greek mathematics with an intimacy that most men have been unable to enjoy, and he has set forth the results of his intimate knowledge with a felicity of expression that could come only from familiarity with the language of the men whose works he has described.

The first difficulty that the historian meets in making known the results of his studies is analogous to the laying out of the ground plan of a structure; it requires him to consider which of several designs he will take and what shall be the order of rooms through which the visitors to the edifice shall be conducted. Moreover, in the case of the history of any science a writer is confronted by one particularly serious difficulty, namely, that of the sequence of chapters. A reader may properly demand that the treatise present the growth of the subject in chronological order, and that it shall also present a subject like analytic geometry as a unit. These two demands are, however, mutually antagonistic, since the chronological order of a history of mathematics would scatter the evidences of the development of any of the leading topics along a period of two or three thousand years, while any mere topical treatment would require a wearisome repetition of chronological and biographical material with every topic considered. Historians are continually trying to harmonize these methods, even as astronomers have tried from time immemorial to harmonize the lunar and solar calendars, and the result in each case is necessarily a compromise. The author has himself called attention to the difficulty, characterizing Professor Loria's *Le Scienze esatte nell' antica Grecia* as "the best history of Greek mathematics which exists at present," and showing that the distinguished Italian historian had taken, as he frankly states, "a compromise between arrangement according to subjects and a strict adherence to chronological order, each of which plans has advantages and disadvantages of its own."

In the work under review the author has made the attempt to solve the problem by arranging his chapters as follows:

I. *Introductory*, the purpose being to give the reader a general view, as from an airplane, of the terrain through which he is to be led—a desirable preliminary treatment in any historical treatise.

II. *Greek numerical notation and arithmetical operations*, setting forth what the reader should know of that part of the numerical art which the Greeks called "logistiké."

III. *Pythagorean arithmetic*, in which is given in considerable detail a survey of that which was known as "arithmetiké"—the theory of numbers.

IV. *The earliest Greek geometry*, in which are set forth the life and labors of Thales. It will be noticed that in Chapters II–IV the author is forced to abandon a strictly chronological sequence, Chapter II considerably overlapping Chapter III, and each overlapping Chapter IV. This is a necessity for a writer who attempts a topical arrangement, and it results in this case in the omission from Chapter III of the contributions of Euclid and Diophantus, for example, these being considered later in chapters devoted to the men themselves.

V. *Pythagorean geometry*, in which the story of geometry is continued, but in which the difficulty is necessarily encountered of distributing the biography of Pythagoras between Chapters III and V. It is interesting to observe that the author has less to say of the life and times of this great philosopher than of any other leading Greek mathematician, probably because less authentic material is available for a biographical sketch. Concerning the much mooted question as to the source of the first demonstration of the theorem which bears the name of Pythagoras, the author gives a judicial summary of the evidence and concludes with the statement: "I would not go so far as to deny to Pythagoras the credit of the discovery of our proposition; nay, I like to believe that tradition is right, and that it was really his"—a decision that will meet with the approval and command the respect of the great majority of students of history.

VI. *Progress in the Elements down to Plato's time*, in which a study is made of one of the most interesting periods in the development of Greek geometry—the formative stage in which proofs were discovered and the logical bases of the science were beginning to be sought. It is now possible for the author to give to the treatise an arrangement that is more nearly biographical, and to set forth the biographies in chronological sequence.

VII. *Special problems*, in which the "three famous problems" of antiquity are considered. Here, as in Chapter VI, the nature of the subject permits of the biographical and chronological treatment. Among other details, Bryson's contributions to the study of the method of exhaustion are recognized more favorably than has of late been the case.

VIII. *Zeno of Elea*. It is probably quite justifiable to give Zeno a chapter by himself, since it would be difficult to place him with anyone else. While the author has not carried his study of the history of the philosophic interpretation of Zeno's problems as far as our Professor Cajori (to whose contribution he pays just tribute), he condenses in a few pages the best of the Greek interpretations of his paradoxes.

IX. *Plato*, in which chapter there is given a succinct statement of the influence of this great philosopher with respect to the foundations upon which a work like Euclid's should rest, this statement being fortified by extracts from Plato's works as well as from those of subsequent writers who were conversant with his doctrine.

X. *From Plato to Euclid*, the period in which the post-Pythagorean accumulation of propositions and the influence of Plato with respect to foundation principles were working towards the making of a treatise which should set forth the Greek geometry in all its excellence. In this period falls the work of Aristotle, not generally enough appreciated in its influence upon geometry but here given just recognition.

XI. *Euclid*, whom no one in our time has recognized so worthily as Sir Thomas Heath, and to whom has here been given about one fourth of Volume I. Naturally the author has done little more than condense into about a hundred pages his own monumental treatise upon the *Elements*, using much of the language there employed, and he could not have done better than follow this plan. Since the belief is not uncommon that Euclid was merely a textbook writer, devoid of mathematical genius, the tribute here paid, showing his genuine powers as a geometer, is welcome. It has been said of Shakespeare that he "took the stillborn children of lesser men's brains and breathed on them the breath of life," and Euclid may have done the same, but it takes a genius, perhaps a divine genius, to perform this miracle.

XII. *Aristarchus of Samos*, more of an astronomer than a geometer, but nevertheless one of the first great geometricians in the astronomical field. It is characteristic of the author that he makes no mention of his own treatise on Aristarchus, indeed, that he hardly refers to any of his other works. This is an illustration of British modesty, to the lack of what they call "side," which we in America (probably unfortunately) fail to understand; for not infrequently the reader might be assisted by more frequent reference to such standard works as those which the author has contributed to the study of Greek mathematics.

XIII. *Archimedes*. It is a tribute to five of the greatest names in the field of ancient mathematical research and exposition that the author has given to each approximately a hundred pages, the total amounting to about half of the entire treatise. These men are Euclid, Archimedes, Apollonius, Pappus, and Diophantus. Of these it is an easy matter to pick out the least, but it is difficult to select the greatest. Probably, if a ballot were taken among those who have the intellectual right to vote, the choice would fall upon Archimedes, and the treatment which the author has accorded him is in harmony with this judgment.

XIV. *Conic Sections: Apollonius of Perga*. As in the cases of Euclid, Aristarchus, Archimedes, and Diophantus, the intellectual world is already familiar with the author's treatises upon this great expositor (and doubtless largely the creator) of the ancient theory of conics. There is no better way of securing an insight into the essential difference between the mathematics of the Greeks and that of the present day than by comparing the treatment, say of the ellipse, as given by Apollonius, with that given in our own modern textbooks, and perhaps there is nothing that gives a student a higher appreciation of the Greek mind. For one who wishes this opportunity but who has but little time for the comparison, this chapter will prove especially helpful.

XV. *The successors of the great geometers*, in which chapter is given a brief statement of the work of those who began, in any large way, the theory of higher plane curves—a theory which Greece had no longer the intellectual strength to complete. The names considered include those of Nicomedes, Diocles, and Perseus—each of whom is known chiefly for a single important contribution to geometry.

XVI. *Some handbooks*, under which properly depreciative title there lie the works of Cleomedes, Nicomachus, and Theon of Smyrna. If ever a man was pushed into fame in the field of mathematics, by little more than chance, that man was Nicomachus.

XVII. *Trigonometry: Hipparchus, Menelaus, Ptolemy*, in this order, for "the first person to make systematic use of trigonometry is, so far as we know, Hipparchus." It was with him that the long union of trigonometry and astronomy began, a union particularly noticeable in the Arab schools, and only broken when each science, in the fifteenth century, had so developed that it was able to stand alone and to set up an establishment for itself.

XVIII. *Mensuration: Heron of Alexandria.* Aside from setting forth a summary of Heron's work the author devotes considerable attention to the controversy which has so long been waged as to the time in which this great scholar lived. Not even Diophantus has given historians so much trouble, partly due to the fact that there were a number of Herons whose names have come down to us together with some knowledge of their achievements. Until recently the Heron of mathematical fame has been commonly placed in the first century B.C. (and sometimes earlier), but of late he has been put in the first century of our era. Heiberg felt that the third century would be a safer conjecture, and Sir Thomas Heath, after carefully weighing the evidence, thinks the same. The evidence is by no means conclusive, but it seems certain that the date was about 50 A.D. or not more than two centuries thereafter.

XIX. *Pappus of Alexandria*, the last of the prominent geometers of Greece; not a great genius but, considering the time in which he lived, a great scholar and a worthy mathematician.

XX. *Algebra: Diophantus of Alexandria*, much more of a genius than Pappus or Heron, a great mathematician in a period of general scientific decay, and the one who best deserves the title of "father of algebra."

XXI. *Commentators and Byzantines.* Under this rather interesting caption, with its hint at mutual exclusion, are considered the names and works of men like Serenus, Theon of Alexandria, Proclus, and Psellus, and of the only prominent woman mathematician of all antiquity—Hypatia. It was the period of the death of Greek science, and, as with all such periods, its chronicles are not stimulating to the mind.

The second volume has a good index and a list of Greek terms.

The limits set for reviews at the present time are such as to allow no adequate statement of the merits of this noteworthy treatise. It is only possible to add that it is destined to be the standard work upon the subject, even as the author's other works are the recognized standards in their respective domains.

The feature which, with respect to the substance of the text, distinguishes the work from any other of its kind is the large amount of source material that it contains. The author has not merely written a history; he has set forth at unusual length the evidence to support his views. Thus we have extracts generously made from such writers as Archimedes, Apollonius, Pappus, and Diophantus by which the reader is able to form a first-hand opinion of the nature and value of their several contributions and methods of attack. Such a plan operates against a literary production of uniform smoothness of expression, and it considers the needs of the scholar instead of the taste of the casual reader; but it is precisely the scholar whom the author has in mind rather than a more general intellectual public for which men like Gibbon and Guizot wrote their justly celebrated treatises. It is difficult to accomplish both purposes; Libri attempted it by giving the scientific world a text composed with true French elegance and supplementing this by source material in an appendix, but in so doing he made the necessary sacrifice of any close connection between his text and his evidence, and moreover his selection of material was rather on the basis of rarity than of importance. On the whole, indeed, no one has solved the problem more effectively than has been done in the work under review.

It should also be said that the plan of quoting so fully from the ancient writers serves to remove any doubt as to the validity of the author's thesis that "the foundations of mathematics and a great portion of its content are Greek. The Greeks laid down the first principles, invented the methods *ab initio*, and fixed the terminology. Mathematics in short is a Greek science, whatever new developments modern analysis has brought or may bring." To establish such a thesis there is demanded not merely the assertions of today; we must have the precise evidence of the past, and this is what has been placed before the reader in such abundance as to make the work a source-book as well as a historical narrative.

If it should be asserted, as the reviewer has already heard it remarked, that the arrangement of material suggests a set of essays rather than a connected historical discourse, the obvious reply is that the history of Greek mathematics is largely a record of the work of a few great geniuses; that it is not, like political history, a list of innumerable wars and changes of dynasties and of perpetual slaughter and rapine, nor even like economic or social history, and that it therefore requires precisely the treatment here accorded to the story of the leaders of scientific thought. It is idle to speculate as to whether the preliminary historical essay might have been expanded into a volume, and the biographical material have been there disposed of first, the special topics being placed in the second volume. Suffice it to say that the author felt that he could accomplish his purpose better by pursuing the route he has taken, and every writer should take the road he can travel most safely. The goal has been reached, the way has been delightful, the guide has spoken as one having authority, and the scientific world should be accordingly generous in its praise and in its appreciation and its thanks.

David Eugene Smith.

SHORTER NOTICES

Lezione di Geometria Proiettiva. Federico Amodeo. Third edition, second reprint. Naples, Luigi Pierro, 1920. 450 pp.

The second reprint of the third edition of this work contains an appendix (43 pp.) of modifications and additions made since the first appearance of the third edition (1905).

The text consists of two parts and each part of three chapters. There is an introductory chapter (66 pp.) preceding Part I in which is set forth the fundamental postulates: definitions of primitive forms; the notions of perspectivity, projectivity, homography, correlation, etc.; the theorem of Desargues and its consequences.

The fundamental postulates are those formulated by Amodeo in ATTI DELL'ACCADEMIA DELLE SCIENZE DI TORINO, March, 1891. The first five deal with the existence of elements, denoted by S_i, $i=0, 1, 2, 3$. The sixth postulate asserts the closure of S_1, considered as a class of S_0's, and thus provides for an ideal point on each line (when $S_0 \equiv$ point, $S_1 \equiv$ line). Postulate seven asserts the invariance of the property of separation of two couples of elements on a primitive form under projection and section. There are two more postulates contained in Chapter I of Part I. The first is equivalent to the Postulate of Archimedes from which is deduced the existence of an infinity of elements of a "succession" (previously defined by harmonic groups) between any two elements of the succession; and the second is the Dedekind Postulate from which is deduced the continuity of each primitive form.

Part I, 217 pages, is devoted to the projective geometry of "simple forms" (primitive forms). It contains Von Staudt's Theorem and its consequences; the theory of projectivities upon primitive forms, of involutions, of polarities in S_2 and S_3; the definition of imaginary elements; etc. Polarities in euclidean space (S_2 and S_3) have centers, diameters, axes, foci, but no explicit mention is here made of conics or cones.

Imaginary elements on any primitive form are defined as double points of an elliptic projectivity upon the form, and are represented by four elements A B C D such that A corresponds to B, B to C, and C to D (A D not separated by B C). This method was given by Amodeo in lectures (lithographed), 1887–88, on homographies in the binary field, and printed in GIORNALE DI BATTAGLINI, volume 26, 1888, in a paper entitled *Fasci di omographie e rappresentazione geometrica degli elementi imaginarii.* Professor J. Rey Pastor, in his book *Fundamentos de la Geometria Proyectiva Superior* (1916), calls this "the method of Amodeo." But the method apparently differs very little from that of Von Staudt, since a pair of conjugate imaginary elements defined as above is also defined by a unique elliptic involution; namely the double-points involution (Involuzione unita, Doppelpunktsinvolution).

Part II, 165 pp., is devoted to forms of the second order and one dimension. These forms are defined by means of projectively related primitive forms rather than as loci of corresponding incident elements in polarities, or as loci corresponding to circles in perspectively related forms, thus following the road chosen by Reye rather than that selected by Von Staudt, or Steiner and Cremona.

The book as a whole is full of detail with good sets of exercises after each chapter. The presentation is clear. The method is synthetic for the most part, although the author does not hesitate to use projective coordinates and the resulting analysis somewhat freely throughout the book. For students who wish to pursue geometry beyond the ordinary undergraduate curriculum in our American universities, Professor Amodeo's book will be both suggestive and inspiring.

L. Wayland Dowling

United States Life Tables 1890, 1901, 1910 and 1901–10. Explanatory Text, Mathematical Theory, Computations, Graphs, and Original Statistics. Also Tables of Life Annuities, Life Tables of Foreign Countries, Mortality Tables of Life Insurance Companies. By James W. Glover. Washington, Government Printing Office, 1921. 496 pp.

The preparation of this volume marks an important advance in the study of vital statistics in the United States.

The work is divided into eight parts. Part I gives a nontechnical description and explanation of life table functions. Part II gives 74 mortality tables based on different subclasses of the population of registration states. Part III gives twelve life tables of foreign countries and ten mortality tables based on experiences of life insurance companies. Part IV presents graphs of life table functions. Part V gives certain life annuities, life insurance premiums and commutation columns. Part VI gives the mathematical theory of the construction of life tables. Part VII gives a detailed account of the process of carrying out the calculations. Part VIII gives tables of the original statistics and an explanation of the different types of data.

The many tables presented in this volume are simply a mine of information for comparative purposes. Part VI on the mathematical theory of the construction of life tables is the part of the work which should prove of much interest to the actuaries interested in the fundamental theory of life table construction. The reviewer believes it correct to say that pages 329–344 give the first clear and detailed statement in the English language of the theory of the continuous flow of population, with applications to the concrete problems of life table construction. The development of these methods centers around the names of Knapp, Zeuner, and Lexis. The author has done an important service in bringing a clear presentation of these methods before American actuaries. In this theory the aggregates of the living and the dead corresponding to assigned birth and age intervals are derived in terms of definite integrals, and the rates of mortality are obtained as the solution of a linear differential equation. The results are made concrete by application to census data.

H. L. Rietz

Espace, Temps, et Gravitation. By A. S. Eddington, translated into French by J. Rossignol, with an introduction by P. Langevin. Paris, J. Hermann, 1921. xii + 262 + 149 pp.

The first part of this volume is a translation of the author's *Space, Time and Gravitation*, reviewed by Professor E. B. Wilson in this BULLETIN (vol. 27 (1921), p. 182). The second part is mathematical and similar to the author's *Report on the Relativity Theory of Gravitation*. The following applies only to the second part.

Eddington has consistently regarded physics as a science where only space-time coincidences or events are observed. These are recorded in terms of coordinates without dimension, then equations are assumed connecting these records and from these equations and records the things of physical significance, such as lengths, times, currents, masses, are *computed*. In this respect he differs from Weyl and most others who regard lengths and times as measured, a procedure which supposes the experimenter endowed with clocks and rods which in some unspecified way adapt themselves to the local geometric requirements.

The exposition is deductive. The postulates include the field equations in empty space and space occupied by matter, and also the differential equations for the motion of a particle. It is possible to derive the field equations by a variation principle where the integrand is the total curvature plus the scalar of the energy tensor. Eddington objects to this because the curvature and energy are physically two aspects of the same thing. On page 76 there is a valuable discussion of the Principle of Equivalence. This is stated in several ways. He uses this principle only a little, "but if we had reasoned by induction, passing from particular laws discovered experimentally to general laws, we would have needed a guiding principle and the Principle of Equivalence would have given us precisely this."

In § 2 there is a much needed treatment of what vectors are in mathematics and of what they are in physics. The last section deals with Weyl's contributions and is characterized by the same vividness and precision as the rest of the book. On page 69, the right-hand member of the second equation of (29, 4) should be negative. This book seems to the reviewer as better adapted than any other to the reader who wishes to spend a limited time on this subject.

K. W. LAMSON

Conférences sur les Transformations en Géométrie Plane. By W. de Tannenberg. Paris, Librairie Vuibert, 1921. 49 pp.

This monograph is concerned with circular transformations, considered primarily as linear transformations of the complex variable. To the beginner it would be of doubtful value, since it assumes the elementary facts concerning the various particular circular transformations and accords scant treatment to the fundamental properties of the general transformation. It deals primarily, both by pure geometry and by analytic methods, with the products of special transformations, e.g., of two rotations or of a rotation and a stretching, and to an advanced student or an instructor interested in this particular subject it should be helpful.

W. C. GRAUSTEIN

Introduction au Calcul Tensoriel et au Calcul Différentiel Absolu. By G. Juvet, with a preface by J. Hadamard. Paris, A. Blanchard, 1922. ii + 101 pp.

This small volume contains a very intelligible account of the so-called "Tensor Theory" which has come to the front with the Einstein relativity. The author starts in with elementary notions of vectors and their linear transformations, then defines tensors algebraically and in connection with bilinear forms, then geometrically. Some consideration of the elementary analysis follows, closing with an exposition of the "parallel displacement" of Levi-Civita, and a general discussion of tensors in a Riemann continuum. There is a good bibliography at the end.

The general point of view is set forth in the interesting preface of Hadamard. His thought reverts back to Poincaré, who said: "The physicist proposes problems to us whose solution he expects. But in making the proposal he pays for the most part in advance for our services. . . . There is an infinite multitude of combinations that may be formed of numbers and symbols. How should one select from this multitude those worthy of attention? Should we be guided by caprice? Such caprice would no doubt alienate our interests far and we would soon cease to understand one another. . . . But there is another side to the matter. . . . Physics not only prevents us from getting lost, but it prevents us from a more serious danger: that of wandering around in a circle."

This situation, Hadamard intimates, had actually arisen in infinitesimal geometry. In fact a crisis had arisen, which fortunately the relativity theory has resolved, through its stimulation of the study of methods already in existence, but mostly ignored, dating back to the absolute geometry of Ricci and Levi-Civita (MATHEMATISCHE ANNALEN, vol. 54).

It is a matter of satisfaction to those of us who have been interested in so-called vector methods, for some time, that they are finally coming into their own. The essential basis of such methods is not the avoidance of coordinates but the development of formulas which are intrinsically given. The present work is open to the criticism to which practically all these investigations are subject, namely that while expressions are produced which are invariant under transformations of the coordinates, there should be no use made of coordinates at all. A proper use of vectors makes the study of absolute geometry not only much simpler and almost obvious, but these same expressions may be translated directly into any desirable system of coordinates with little trouble. The invariancy is a direct consequence of the method. The author remarks in his introduction: "Systems of coordinates are not rejected, and in place of being foreign to the things studied, actually form their structure." Such a point of view is the common one, it is freely admitted, but it is to the detriment of the things studied. It merely shows the path of least resistance taken by the minds of the investigators. When a new generation think as readily in general vectors as many now do in ordinary vectors of three-dimensional space, instead of long demonstrations of invariant and covariant forms, occupying many pages, there will be a few pages of direct statements, which reach the heart of the matter at once. The real criticism on the use of vectors is that they practically demand that the curved space under consideration be embedded in a flat space of a requisite number of dimensions. This difficulty, however, seems to exist as well for those methods that insist on coordinates. The work of M. Juvet is an example. A perfect intrinsic treatment would never leave the space itself. This can be accomplished by a properly developed system of general vectors. However, when one realizes that if the curved space is embedded in a flat space, then every formula in terms of vectors in the flat space relating to the curved space is *ipso facto* a covariant formula, he will see the whole matter of "theory of tensors" in the proper light.

Hadamard is certainly correct in taking the position that whatever the value of the relativity theory may be, it has done a great thing for geometry in opening up a new life to it, not of the momentary character of a new attack by some geometer, but of the permanent character infused by nature herself. Not only theorems in physics should be stated without systems of axes for reference, but theorems in geometry should also be so stated. The present work will be welcome to those who do think in coordinates, for its clarity and simple presentation.

JAMES BYRNIE SHAW.

Annuaire du Bureau des Longitudes pour l'An 1922. Paris, Gauthier-Villars. 1922. 7 + 800 pp.

There is little to say concerning the current issue of the ANNUAIRE. The review copy arrived somewhat late in the year but this matters less than might be imagined for an almanac, since the data, chiefly astronomical, which change from year to year, are always given in the almanac of the previous year.

There is an attractive little article on Relativity by E. Picard which sets forth the principal points of the theory and the astronomical tests. An article on Money and Exchange by Ch. Lallemand explains the fundamental bases of past and present currencies, and shows the fluctuations of their gold values in some detail during the past eight years.

E. W. BROWN.

Kartenkunde. By M. Groll. Neu bearbeitet von Dr. Otto Graf. I. *Die Projektionen.* Berlin und Leipzig, Vereinigung wissenschaftlicher Verleger, 1922. 116 pp.

This short treatise on map-projections is No. 30 of the well known Sammlung Göschen, and gives a fairly complete account of the numerous systems of mapping of the terrestrial globe. The introduction, which is concerned with general information about drawing, scales, and drawing instruments, and physical geography, is succeeded by four chapters dealing with various projections of the spherical surface upon a plane. In the fifth and last chapter we find a valuable summary and illustrations of the various methods of mapping in use and an historical sketch of their development.

On the whole the little book will be appreciated by students who wish to acquire a general knowledge of map-projections without spending too much time in studying more ambitious and special treatises on the subject. ARNOLD EMCH.

A Study of Mathematical Education. By Benchara Branford. New edition, enlarged and revised. Oxford, at the Clarendon Press, 1921. xii + 420 pp.

Comparison with the 1908 edition shows the book under review to be a reprint rather than a completely revised edition; according to the preface "the changes in the original, though numerous, are in general of subordinate details," which changes are certainly minor judging from the obvious identity between the two editions even in the indexes. The last 40 pages, or Part III, are entirely new to this edition and under the caption "The Past, the Present, and the Future" discuss the subjects "Adolescent Technique," "Specialists and Cosmology," "Algebra and Mensuration," and "Comparative Algebra, Geometry and Mechanics."

Even though containing little new matter, "the world-wide welcome generously given the work, including a pre-war German translation and a Russian translation in course of preparation," seems to justify this reprint. The work may be characterized as the interesting and thought-provoking meditations on mathematical education of a teacher of twenty or more years' experience in nearly all grades of English schools. The reader feels some lack of unity in the book as a whole which is no doubt due to the fact that it is largely a collection of articles, courses of lectures and addresses written at various times and for various purposes. The following topics and phrases may give some idea of the type and range of the discussions: experimental mathematics, measurement in geometry, types of evidence, educational principles, suggestions from historical developments, nature of geometric knowledge, culture and occupation, presentation determined by ability and background of the learner, good and bad results of the dethronement of Euclid, degree of rigor best fitted to the maturity of the learner, the tongue-tied practical man, danger of discontinuity of logical treatment, need of smaller logically developed systems of propositions, relations of algebra and geometry, suggestions to teachers from non-euclidean geometries and the Einstein theories, the great unities of mathematics, teaching principles as ideals. The progressive teacher sensitive to new ideas and new viewpoints as suggesters in his own thinking and planning will profit by reading this book. ERNEST B. LYTLE

Lezioni di Statistica Metodologica. By Filadelfo Insolera. Turin, Libreria Fratelli Treves, 1921. 191 pp.

This small volume deals with a large variety of topics, including approximate computation, averages, measures of dispersion, permutations, combinations, probability, binomial distribution of frequency, interpolation by the formulas of Newton and Lagrange, graduation of data, least squares, moments, correlation and contingency. The book gives brief elementary expositions of these topics, and will probably serve well its purpose as a means of preparation for certain examinations. On account of the lack of illustrative examples, the reviewer is of the opinion that the book would not be a suitable substitute for certain English and German books for the beginner in the study of statistics. H. L. RIETZ

NOTES

At the meeting of the American Association for the Advancement of Science held at Cambridge, December 26–30, 1922, Professor Harris Hancock of the University of Cincinnati was elected vice-president of Section A, Professor G. A. Miller a member of the Council, and Professor Frank Schlesinger a member of the Committee on Grants for Research in charge of mathematics and astronomy. Professor E. H. Moore delivered his address as retiring president, on the subject, *What is a number system?* Professor Oswald Veblen's address, as retiring vice-president of Section A, was entitled *Geometry and physics,* and was delivered before a joint session of Section A, the American Mathematical Society, and the Mathematical Association of America.

The April, 1922, number (vol. 44, No. 2) of the AMERICAN JOURNAL OF MATHEMATICS contains: *A primary classification of projective transformations in function space*, by L. L. Dines; *A general theory of limits*, by E. H. Moore and H. L. Smith; *Substitution groups whose cycles of the same order contain a given number of letters*, by G. A. Miller; *Boundary value and expansion problems: oscillation, comparison and expansion theorems*, by R. D. Carmichael; *On a theorem in general analysis and the interrelations of eight fundamental properties of classes of functions*, by E. W. Chittenden.

The American Telephone and Telegraph Company has begun publication of a new periodical, THE BELL SYSTEM TECHNICAL JOURNAL, devoted to the scientific and engineering aspects of electrical communication. The two numbers that have appeared contain articles on the mathematical as well as the experimental physics of the telephone and telegraph.

The first number of a new Italian mathematical journal, the BOLLETTINO DELLA UNIONE MATEMATICA ITALIANA, was issued under the date of October, 1922. Professor S. Pincherle, of Bologna, is provisional president of the union.

The Paris Academy of Sciences announces the award of the following prizes for 1922, in addition to those listed in the December number of this BULLETIN (p. 477). Since no memoir was presented on the subject announced for the Grand prize, *The determination of classes of surfaces by given properties of their geodesics* (see this BULLETIN, vol. 26, p. 282), this subject was withdrawn, and the prize awarded to Professor Jean Le Roux, of the University of Rennes, for his mathematical work. The Poncelet prize was awarded to Professor Jules Drach, of the University of Paris, or his mathematical work; the Francœur prize to Dr. Louis Antoine, of the University of Strasbourg, for his work in geometry; the Montyon prize in mechanics to Farid Boulad, of the Egyptian State Railways, for his work in nomography and geometry; the Henri de Parville prize in mechanics to Henri Béghin, of the University of Montpellier, for his memoir on gyroscopic compasses; the Valz prize to Professor Jean Chazy, of the University of Lille, for his work in celestial mechanics and especially his memoir on the problem of three bodies; the Hughes prize to Camille Raveau, for his work in theoretical physics; the Montyon prize in statistics to Pierre Richard, for his work on the mathematical theory of insurance; the Binoux prize in the history and philosophy of sciences to Professor Gino Loria, for his works on the history of sciences; the Houllevigue prize to Professor Rodolphe Soreau, for his work in aviation and nomography.

The honorary doctorate of the University of Paris has been conferred on Professor A. A. Michelson, of the University of Chicago.

Professor Charles Camichel, of the University of Toulouse, has been elected corresponding member of the Paris Academy of Sciences, in the section of mechanics.

Professor Albert Einstein has been elected a member of the Russian Academy of Sciences.

Professor C. G. Darwin, who is serving as professor of mathematical physics at the California Institute of Technology for the year 1922–23, has been appointed to the newly instituted Tait Chair of Natural Philosophy at the University of Edinburgh.

Associate Professor Emma L. Konantz, of Ohio Wesleyan University, has accepted a permanent appointment at Peking University, effective the second semester of this year. Miss Konantz taught at Peking University while on leave of absence in 1919–21.

Dr. V. E. Pound, of the University of Toronto, has been appointed instructor in mathematics at the University of Buffalo.

Professor G. D. Birkhoff, of Harvard University, will deliver a course of lectures in the graduate school of Yale University during the second half of the present academic year, in the absence of Professor E. W. Brown.

At Columbia University, Professor T. S. Fiske, of the department of mathematics, and Professor C. L. Poor, of the department of celestial mechanics, have been granted leave of absence for the second half year.

Professor Oswald Veblen, of Princeton University, president of the American Mathematical Society, has been granted leave of absence for the second half year.

Professor B. F. Dostal, of the University of Denver, has been appointed professor of mathematics at the Bradley Polytechnic Institute.

Miss Eleanor P. Cushing has been made professor emeritus at Smith College.

NEW PUBLICATIONS

I. PURE MATHEMATICS

BRILL (A.). See SALMON (G.).

FIEDLER (W.). See SALMON (G.).

GALBRUN (H.). Introduction à la théorie de la relativité. Calcul différentiel absolu et géométrie. Paris, Gauthier-Villars, 1923. 8vo 10 + 459 pp.

GRIGORAS (E. C.). Sur la relativité en algèbre. Bucarest, C. Jonesco, 1922.

HARPER (H. D.). See WENTWORTH (G.).

HEFFTER (L.). Was ist Mathematik? Unterhaltungen während einer Seereise. Freiburg i. Br., Theodor Fischer, 1922. 160 pp.

HUMBERT (E.). Traité d'arithmétique. 5e édition. Paris, Vuibert, 1922. 8vo. 508 pp.

KOMMERELL (K.). See SALMON (G.).

MICHAELIS (L.). Einführung in die Mathematik für Biologen und Chemiker. 2te, erweiterte und verbesserte Auflage. Berlin, Springer, 1922. 6 + 318 pp.

MIRINNY (L.). Isonolyse. (Résolution générale des équations.) Intégration et résolution générale des équations différentielles. Paris, Imprimerie Chaix, 1922. 12 pp.

SALMON (G.) und FIEDLER (W.). Analytische Geometrie des Raumes. Unter Mitwirkung von A. Brill neu herausgegeben von K. Kommerell. 1ter Teil: Die Elemente und die Theorie der Flächen zweiter Ordnung. 2te Lieferung. 5te Auflage. Leipzig, Teubner, 1923. 4+369–612 pp.

SCHEFFERS (G.). See SERRET (J. A.).

SCHUSTER (A.). Mathematische Unterrichts-Briefe. Band 2: Uebungsbuch. Berlin, Vereinigung Wissenschaftlicher Verleger, 1922. 8vo. 4 + 288 pp.

SERRET (J. A.) und SCHEFFERS (G.). Lehrbuch der Differential- und Integralrechnung. 6te und 7te Auflage. Band 2: Integralrechnung. Leipzig, 1921.

SMITH (D. E.). See WENTWORTH (G.).

STRUIK (D. J.). Grundzüge der mehrdimensionalen Differentialgeometrie in direkter Darstellung. Berlin, Springer, 1922. 8 + 198 pp.

WENTWORTH (G.), SMITH (D. E.), and HARPER (H. D.). Fundamentals of practical mathematics. Boston, Ginn, 1922. 5 + 202 pp. $1.50.

YULE (G. U.). An introduction to the theory of statistics. 6th edition, enlarged. London, C. Griffin, 1922. 8vo. 430 pp.

II. APPLIED MATHEMATICS

Brose (H. L.). See Moskowski (A.).

Dufrenois (—.), Risser (—.), et Rousier (—.). Les méthodes actuelles de la balistique extérieure. Paris, Gauthier-Villars, 1921. 10 + 244 pp.

Duval (A. B.) et Hébrard (L.). Traité pratique de navigation aérienne. Paris, Gauthier-Villars, 1922. 4to. 60 pp.

Eddington (A. S.). Exposé théorique de la relativité généralisée. Complément mathématique inédit de l'édition française d'Espace, temps et gravitation. Paris, Hermann, 1922. 4to. 4 + 149 pp.

Einstein (A.). Untersuchungen über die Theorie der "Brownschen Bewegung." Mit Anwerkungen herausgegeben von R. Fürth. (Ostwald's Klassiker der exakten Wissenschaften.) Leipzig, Akademische Verlagsgesellschaft, 1922. 8vo. 72 pp.

———. See Lorentz (H. A.).

Encyklopädie der mathematischen Wissenschaften. Band V 1, Heft 6: K. F. Herzfeld, Physikalische und Elektrochemie. Leipzig, Teubner, 1921.

Exner (F.). Vorlesungen über die physikalischen Grundlagen der Naturwissenschaften. 2te Auflage. Leipzig und Wien, Deuticke, 1922. 20 + 734 pp.

Fabry (C.). Eléments d'électricité. Paris, Colin, 1921. 208 pp.

Fisher (I.). The making of index numbers. A study of their varieties, tests, and reliability. Boston, Houghton Mifflin, 1922. 32 + 526 pp.

Föppl (A.). Vorlesungen über technische Mechanik. Band 5 und 6. 4te Auflage. Leipzig, 1921–1922.

Fürth (R.). See Einstein (A.).

Gandillot (M.). Véritable interprétation des théories relativistes. Paris, Gauthier-Villars, 1922. 16mo. 17 pp.

Gibson (A. H.). Hydro-electric engineering. Volume 1: Civil and mechanical engineering. London, Blackie, 1921. 10 + 232 pp.

Graetz (L.). Handbuch der Elektrizität und des Magnetismus. In fünf Bänden. Herausgegeben von L. Graetz. Leipzig, Barth. Band 1: Elektrizitätserregung und Elektrostatik. 1918. 8 + 760 pp. Band 2: Stationäre Ströme. 1921. 10 + 772 pp. Band 4: Magnetismus und Elektromagnetismus. 1920. 10 + 1360 pp.

Grimsehl (E.). Lehrbuch der Physik. Band 1: Mechanik, Wärmelehre, Akustik und Optik. 5te Auflage. Leipzig, Teubner, 1921. 16 + 1029 pp.

Gruner (P.). Leitfaden der geometrischen Optik und ihrer Anwendungen auf die optischen Instrumente. Bern, 1921.

Guilbert (G.). La prevision scientifique du temps. Paris, Challamel, 1922.

Hauber (W.). Statik. Teil 1: Die Grundlehren der Statik starrer Körper. 6ter Neudruck. Berlin, 1921.

Hébrard (L.). See Duval (A. B.).

Helmholtz (H. von). Vorlesungen über theoretische Physik. Band 6:

Vorlesungen über die Theorie der Wärme. 2ter, durchgesehener Abdruck. Leipzig, Barth, 1922. 12 + 418 pp.

Henning (F.). See Seeliger (R.).

Hentschel (W.). Das Relativitätsprinzip in Rahmen einer Gesamtansicht von Welt und Mensch. Leipzig, E. Matthes, 1921. 8vo. 77 pp.

Herzfeld (K. F.). See Encyklopädie.

Huguet (L.) et Suffrin-Hébert (M.). Calculs aéro-dynamique des avions. Lois de résistance de l'air. Paris, Béranger, 1922. 16mo. 120 pp.

Kirchberger (P.). Die Entwicklung der Atomtheorie. Karlsruhe, Müller, 1922. 8vo. 10 + 260 pp.

Kossel (W.). Valenzkräfte und Röntgenspektren. Zwei Aufsätze über das Elektronengebäude des Atoms. Berlin, Springer, 1921. 70 pp.

Langevin (P.). See Thompson (J. J.).

Lebel (J.). Résumé des principes de la perspective linéaire. Paris, Vuibert, 1921. 48 pp.

Lemoine (J.). Précis de physique à l'usage des élèves de mathématiques spéciales. Tome 1: Optique. Paris, Vuibert, 1922. 270 pp.

Leslie (G. H.). See Thompson (G.).

Lorentz (H. A.), Einstein (A.), Minkowski (H.). Das Relativitätsprinzip. 4te, vermehrte Ausgabe. Leipzig, Teubner, 1922.

Mallik (D. N.). The elements of astronomy. Cambridge, University Press, 1921. 8 + 233 pp.

Märtens (F.). Die Welt als Wirkung strahlender Energien. Heft 3. Elberfeld, Selbstverlag, 1922. 64 pp.

Minkowski (H.). See Lorentz (H. A.).

Mises (R. von). Naturwissenschaft und Technik der Gegenwart. Eine akademische Rede mit Zusätzen. Leipzig, Teubner, 1922.

———. See Seeliger (R.).

Moch (G.). Initiation aux théories d'Einstein. Paris, Librairie Larousse, 1922. 12mo. 160 pp.

———. La relativité des phénomènes. Paris, E. Flammarion, 1921. 18mo. 366 pp.

Moskowski (A.). Einstein. Einblick in seine Gedankenwelt. Berlin, F. Fontane, 1922. 240 pp.

———. Einstein the searcher; his work explained from dialogues with Einstein. Translated by H. L. Brose. London, Methuen, 1921. 12 + 246 pp.

Mottelay (P. F.). Bibliographical history of electricity and magnetism, chronologically arranged. Researches into the domain of the early sciences, especially from the revival of scholasticism, with biographical and other accounts of the most distinguished natural philosophers throughout the middle ages. Compiled by P. F. Mottelay. London, C. Griffin, 1922. 8vo. 20 + 673 pp.

Otzen (R.). Praktische Winke zum Studium der Statik und zur Anwendung ihrer Gesetze. 3te Auflage. Berlin, Springer, 1921.

PALMER (A. R.). The use of graphs in commerce and industry. London, Bell, 1921. 9 + 47 pp.

PERCIVAL (A. S.). Perspective. London, Longmans, 1921. 42 pp.

PETZOLDT (J.). Das Weltproblem vom Standpunkte des relativistischen Positivismus aus, historisch-kritisch dargestellt. 3te, neubearbeitete Auflage, unter besonderer Berucksichtung der Relativitätstheorie. (Wissenschaft und Hypothese, Band 14.) Leipzig, Teubner, 1921.

PLANCK (M.). Einführung in die allgemeine Mechanik. Zum Gebrauch bei Vorträgen sowie zum Selbstunterricht. 3te Auflage. Leipzig, Hirzel, 1921. 7 + 226 pp.

PLANCK (M.). Vorlesungen über die Theorie der Wärmestrahlung. 4te Auflage. Leipzig, Barth, 1921. 11 + 224 pp.

RAMAN (C. V.). Molecular diffraction of light. Calcutta, University of Calcutta, 1922. 8vo. 10 + 103 pp.

RISSER (—.). See DUFRENOIS (—.).

ROUSIER (—.). See DUFRENOIS (—.).

SCHLICK (M.). Raum und Zeit in der gegenwärtigen Physik. 4te vermehrte und verbesserte Auflage. Berlin, Springer, 1922. 6 + 108 pp.

SCHMID (T.). Darstellende Geometrie. Band I. 3te Auflage. Berlin, Vereinigung Wissenschaftlicher Verleger, 1922. 283 pp.

SEELIGER (R.). Aufgaben aus der theoretischen Physik. In Verbindung mit F. Henning und R. von Mises herausgegeben. Braunschweig, Vieweg, 1921. 9 + 154 pp.

SEILIGER (M.). Graphische Thermodynamik und Berechnung der Verbrennungsmaschinen und Turbinen. Berlin, Springer, 1922. 8 + 250 pp.

SILBERSTEIN (L.). The theory of general relativity and gravitation. Toronto, University of Toronto Press, 1922. 8vo. 6 + 141 pp.

SMITH (L. R.). Industrial physics: Mechanics. New York and London, McGraw-Hill, 1922. 11 + 226 pp.

SOLOVINE (M.). See THOMPSON (J. J.).

SUFFRIN-HÉBERT (M.). See HUGUET (L.).

THOMPSON (G.) and LESLIE (G. H.). Heat. Part 1. London, Cassell, 1922. 8vo. 160 pp.

THOMSON (J. J.). Electricité et matière. Traduction de M. Solovine. Préface de P. Langevin. Paris, Gauthier-Villars, 1922. 12mo. 10 + 132 pp.

TUTTON (A. E. H.). Crystallography and practical crystal measurement. In 2 volumes. 2d edition. London, Macmillan, 1922. 17 + 746 + 14 + 8 + 700 pp.

WEISER (M.). Das Atom: eine gemeinverständige Darstellung der neueren Ergebnisse der physikalischen Strahlenforschung. Dresden, Pahl, 1922. 8vo. 64 pp.

WICHERS (A. W.). De rechtsleer der levensverzekeringsovereenkomst. Aflevering 5. The Hague, Nijhoff, 1922. 64 pp.

THE TWENTY-NINTH ANNUAL MEETING OF THE SOCIETY

The twenty-ninth annual meeting of the Society was held at Harvard University on Wednesday and Thursday, December 27–28, 1922, during the period of the Convocation week meetings of the American Association for the Advancement of Science and immediately preceding the meeting of the Mathematical Association of America. Dormitory accommodations were furnished by Harvard University and Radcliffe College, and Phillips Brooks House was used as a social center for the mathematical group. Not only was there a record attendance at the meeting, but members found it stimulating and pleasant to come into contact with scientists working in fields related to mathematics. At the close of the sessions it was voted to express the thanks of the Society to Harvard University for its hospitality and to the departments of mathematics at both Harvard and the Massachusetts Institute of Technology for the admirable way in which arrangements for the meetings were carried out.

Members of the Society were invited to hear the address of Professor E. H. Moore, retiring president of the American Association for the Advancement of Science, on *What is a number system?*, delivered Tuesday evening at the Massachusetts Institute of Technology. Joint sessions were held on Wednesday afternoon with Section A of the American Association for the Advancement of Science, at the Massachusetts Institute of Technology, and on Thursday afternoon with the Mathematical Association of America and Section A at Harvard University. An enjoyable occasion was the joint dinner of the three organizations held at the Massachusetts Institute of Technology on Thursday evening.

The attendance included the following 148 members of the Society:

Alexander, E. S. Allen, Archibald, Atchison, Bailey, Beatley, W. J. Berry, Bill, Birkhoff, Bliss, Borden, Bowler, B. H. Brown, E. P. Brown, Bryan, M. Buchanan, W. G. Bullard, R. W. Burgess, Burkett, Burwell, Cairns, Cajori, B. H. Camp, Carr, B. E. Carter, Clawson, Coates, Coolidge, Copeland, Crum, Currier, Denton, Dimick, Dodd, Eisenhart, L. C. Emmons, Esty, G. C. Evans, G. W. Evans, Falconer, Floyd Field, Fields, C. H. Forsyth, Fort, Philip Franklin, Frizell, Garabedian, Gerst, Gilman, Gleason, Glenn, Glover, Gould, M. C. Graustein, W. C. Graustein, Gummer, Hammond, J. G. Hardy, Haskins, Hazlett, Hicks, Hille, Hodgkins, Huntington, Hurwitz, Ingraham, Kazarinoff, Kellogg, Kennelly, W. D. Lambert, Larew, Leib, Leyzerah, Lipka, McGavock, MacDuffee, MacPherson, H. P. Manning, Marriott, G. A. Miller, Molina, C. L. E. Moore, C. N. Moore, E. H. Moore, R. K. Morley, H. C. M. Morse, F. R. Moulton, F. H. Murray, Musselman, Newkirk, Olds, Osgood, Packer, Pattillo, Pell, Perkins, Phillips, Pitcher, Ransom, Ranum, L. H. Rice, R. G. D. Richardsons, H. L. Rietz, E. D. Roe, Roever, Ruger, Rutledge, Safford, Seely, Sharpe, Shaub, Shewhart, Silverman, W. G. Simon, Sinclair, Slaught, Slobin, C. E. Smith, D. E. Smith, S. E. Smith, M. J. Sperry, Stetson, Swartzel, J. S. Taylor, J. M. Thomas, Tracey, Tyler, Van Vleck, Veblen, Vivian, Waddell, J. L. Walsh, Webster, Wedderburn, Weisner, M. E. Wells, H. S. White, D. E. Whitford, Whittemore, Wiener, C. E. Wilder, F. B. Williams, Willis, R. G. Wood, F. S. Woods, J. M. Young, J. W. Young, Zeldin.

President Bliss presided at the session Wednesday morning, relieved by Professor Veblen. President Veblen presided at the joint session on Wednesday afternoon, Professor Osgood on Thursday morning, and Professor G. A. Miller, Vice-President of Section A, at the joint session on Thursday afternoon.

At the meeting of the Council, the following twenty-two persons were elected to membership in the Society:

Professor Ralph Beatley, Harvard Graduate School of Education;
Mrs. Ethelwynn Rice Beckwith, Harvard University;
Rev. Dr. Martin Brennan, St. Louis, Mo.;
Professor James Bruce Coleman, University of South Carolina;
Professor Marion Alonzo Eason, United States Naval Academy;
Mr. Hervey Crandall Hicks, Brown University;
Mr. William Charles Janes, Kansas State Agricultural College;
Mr. Carl Otto Lampland, Lowell Observatory;
Mr. Frank Solimena Langellotti, New York, N. Y.;
Miss Martha Peirce McGavock, Wellesley College;
Mr. Albert Eugene Meder, Columbia University;
Dr. Winifred Edgerton Merrill, Mamaroneck, N. Y.;
Dr. Harvey Pierson Pettit, University of Illinois;
Mr. Pincus Schub, Hebrew School of Newark;
Professor Charles Cutler Spooner, Northern State Normal School, Marquette, Mich.;
Mr. Charles Aynes Stanwick, Western Electric Company;

Professor Tsurusaburo Takasu, Tôhoku Imperial University;
Dr. Louis Ten Eyck Thompson, Kalamazoo College;
Dr. Elihu Thomson, General Electric Company;
Professor Edward Lee Thorndike, Teachers College, Columbia University;
Miss Virginia Watts, New Orleans, La.;
Mr. Frederick W. Weston, Wynnewood, Pa.

Twenty-five applications for membership were received. The Secretary announced that the following members of the London Mathematical Society had entered the American Mathematical Society in consequence of the reciprocity agreement recently arranged between the two Societies:

Professor Mathew J. Conran, University College, Cork;
Mr. Krishna Prasad Dè, University College, Rangoon;
Professor Arthur Lee Dixon, Magdalen College, Oxford;
Principal John Leigh Smeathman Hatton, East London College, University of London;
Professor Harold Hilton, Bedford College, University of London;
Sir Joseph Larmor, St. John's College, Cambridge;
Professor George Henry Livens, University College, Cardiff;
Professor William Proctor Milne, University of Leeds;
Professor Archibald Read Richardson, University College, Swansea;
C. M. Ross, King's College, Cambridge;
Mr. Reginald Owen Street, University of Liverpool;
Mr. Bertram Martin Wilson, University of Liverpool.

The Council announced that the invitation of Vassar College to hold the summer meeting of 1923 at that institution had been accepted and that the annual meeting would be held in New York City.

The committee on membership made a final report through its chairman, President Bliss. Since October 1921 the committee by enlisting the cooperation of several members of the Society has obtained some 228 new members, 33 new subscriptions to the TRANSACTIONS and $372.00 in gifts from members to the regular funds. It recommended that in view of the still critical state of the finances of the Society a new committee on membership be appointed. In discharging the committee, the Council thanked the chairman for the splendid service rendered to the Society. A new committee on membership was appointed consisting of Professors Abraham Cohen, W. H. Roever, and C. E. Smith (permanent secretary).

Professor G. A. Bliss was elected a member of the International Mathematical Union to fill the vacancy caused by Professor Veblen's succession to the presidency of the Society. Professors C. N. Moore and R. G. D. Richardson were appointed to represent the Society on the Council of the American Association for the Advancement of Science for the year 1923.

The Council authorized the appointment by the President of standing committees on endowment, and on policy and budget.

The total membership of the Society is now 1127, including 82 life members. The total attendance of members at all meetings, including sectional meetings, during the past year was 538; the number of papers read was 228. The number of members attending at least one meeting was 377. At the annual election 246 votes were cast.

The reports of the Treasurer and the Auditors (Mr. S. A. Joffe and Professor W. J. Berry) were received, showing a balance $13,232.36, exclusive of special funds; of this $5782.85 is reserved to secure the life memberships. It was explained that the large balance on hand is due to the fact that the Transactions of the Society is eighteen months late in appearing. Sales of the Society's publications during the year amounted to $3,298.43.

The Library now contains 6,203 volumes, exclusive of members' dissertations.

At the annual election, which closed on Wednesday morning, the following officers and other members of the Council were chosen:

President, Professor OSWALD VEBLEN.

Vice-Presidents, Professor R. L. MOORE, Professor H. W. TYLER.

Secretary, Professor R. G. D. RICHARDSON.

Treasurer, Professor W. B. FITE.

Librarian, Professor R. C. ARCHIBALD.

Committee of Publication

Professors E. R. HEDRICK, W. A. HURWITZ, J. W. YOUNG.

Members of the Council to serve until December, 1925

Professor E. T. Bell, Professor J. C. Fields,
Professor W. H. Bussey, Professor Arthur Ranum.

The amendment to the constitution designating as permanent member of the Council any ex-Secretary who had served ten years was carried.

The joint session on Wednesday afternoon was devoted to a symposium on *Space and Time* and was attended by about three hundred scientists. The following papers were read:

I. *The logic of space and time,* by Professor G. D. Birkhoff.

II. *The physical meaning of space and time,* by Professor P. W. Bridgman.

III. *The astronomical measures of space and time,* by Professor Harlow Shapley.

The program of the joint session on Thursday afternoon was as follows:

I. *The reduction of singularities of plane curves by birational transformation,* by Professor G. A. Bliss, retiring President of the Society. This paper will appear in full in the April number of this Bulletin.

II. *The grafting of the theory of limits on the calculus of Leibniz,* by Professor Florian Cajori. (Address delivered at the request of the Mathematical Association of America.)

III. *Geometry and physics,* by Professor Oswald Veblen, retiring Vice-President of Section A.

Titles and abstracts of the papers read at the regular sessions of the Society follow below. Dr. Bray's paper was read by Professor Evans. The papers of Professors Graustein, C. L. E. Moore, Craig, Cajori, Bell, R. L. Moore, and Camp, Dr. Post, Professor Schwatt, Mr. Poritsky, Professors Hitchcock and Cowley, Mr. Murray, Professor Alexander, and Mr. Raynor, and the second papers of Dr. Walsh, Dr. B. H. Brown, Dr. Wiener, Dr. Franklin, Professor Coolidge, Professor Miller, and Dr. MacDuffee were read by title. Mr. Brinkmann and Mr. Linfield were introduced by Professor Birkhoff and Professor Hitchcock by Professor C. L. E. Moore.

1. Dr. J. L. Walsh: *On the location of the roots of polynomials.*

It is the purpose of this paper to prove the following theorem: Let circles $C_1, C_2, \cdots, C_k$, whose centers are the points $\alpha_1, \alpha_2, \cdots, \alpha_k$, be the loci of $n_1, n_2, \cdots, n_k$ roots of a polynomial $f(z)$ which has no other roots, where the circles C_i are all equal and their centers α_i all lie on a line parallel to the axis of reals. If these circles are sufficiently small, and if the polynomial

$$a_0z^n + na_1z^{n-1} + n(n-1)a_2z^{n-2} + \cdots + n(n-1)\cdots 2\cdot 1a_n, \qquad (n = n_1 + n_2 + \cdots + n_k),$$

has only real roots, then the locus of the roots of the polynomial $F(z) = a_0f(z) + a_1f'(z) + a_2f''(z) + \cdots + a_nf^{(n)}(z)$ consists of the circles C_j' that are equal to the circles C_i and whose centers are the roots of $F(z)$ when the roots of $f(z)$ are the points $\alpha_1, \alpha_2, \cdots, \alpha_k$ of multiplicities $n_1, n_2, \cdots, n_k$ respectively. Any circle C_j' which has no point in common with any of the other circles C_m' contains a number of roots of $F(z)$ equal to the multiplicity of its center as a root of $F(z)$ when the roots of $f(z)$ are the points α_i.

2. Dr. J. L. Walsh: *A generalization of normal congruences of circles.*

This paper appeared in the December BULLETIN.

3. Dr. Eleanor P. Brown: *Expansion theorems for a certain homogeneous integral equation.*

This paper deals with a Fredholm linear homogeneous integral equation $u(x) = -\rho^2\int_\alpha^\beta K(x, \xi)u(\xi)d\xi$, whose kernel is such that, though continuous and possessing continuous derivatives on either side of the line $x = \xi$, in the x, ξ plane, it has certain "finite jumps" in its derivatives as the point (x, ξ) crosses this line. Provided certain orthogonality relations are not satisfied, the equation can be reduced to a differential equation of Sturm-Liouville type with the addition of certain integral terms, along with a pair of boundary conditions. These integral terms have little effect on the solutions of the equation, and it is shown that the characteristic numbers and functions behave essentially like those of the corresponding Sturm-Liouville system. The fundamental theorem is proved that if we set up the formal Fourier series, with respect to each of the two above-mentioned systems, of any function $\Phi(x)$ which, in absolute value, is integrable, then the difference (taken termwise) between these two series converges absolutely and uniformly to zero.

4. Dr. B. H. Brown: *The equilong transformations of euclidean space.*

The equations of the equilong transformations of euclidean space, in Bonnet oriented tangential coordinates, are

Direct: $U = U(u),\ \ V = V(v),\ \ W = \sqrt{U'V'}\,w + F(u, v);$

Indirect: $U = U(v),\ \ V = V(u),\ \ W = \sqrt{U'V'}\,w + F(u, v).$

The forms given by Blaschke (ARCHIV, 1910), and all subsequent references in the literature contain a ± sign under the radicals, which is incorrect.

The main portion of this paper deals with the invariance of differential expressions and equations under these, or subgroups of these transformations. Applications are made to one-sided surfaces, to minimal surfaces and their associated surfaces, and to a large class of surfaces defined by second order linear partial differential equations, including the surfaces of Appell, of Goursat, and of Bonnet. The following theorem is proved: Any oriented non-developable surface may, by each of two and only two distinct, direct, equilong transformations, be transformed into any other oriented non-developable surface, and that with an arbitrary analytic directly conformal mapping of their spherical representations. This theorem was suggested by Study, but he stated, incorrectly, that there was one and only one such transformation.

5. Dr. B. H. Brown: *Contact transformations linear in x, y, and z; applications to equilong transformations.*

A transformation of Lie planar elements, in which X, Y, and Z are linear in x, y, and z, carries planes into planes if and only if all the three-rowed minors of a certain ten-by-three matrix vanish. Necessary and sufficient conditions for equilong transformations, a subgroup of the preceding group, follow easily. For a particular subgroup of the equilong group, these conditions reduce to two partial differential equations which are necessary and sufficient conditions that the parameters u and v of a general surface $x = x(u, v)$ have the geometric significance of p and q. Certain rational solutions of these equations furnish the parametric equations of well known one-sided surfaces. The transformation of Bonnet is shown to be equilong.

6. Professor W. C. Graustein: *On the determination of a surface by its principal normal curvatures and spherical representation.*

It is shown in this paper that a non-developable surface is, in general, uniquely determined, to within its position in space, by the linear element of its spherical representation and its principal normal curvatures, and that the exceptional cases, for which definitive conditions are given, lead either to two or to a single infinity of non-congruent surfaces.

7. Professor W. C. Graustein and Mr. B. O. Koopman: *A necessary and sufficient condition that a map of two surfaces be isometric.*

It is well known that in order that a map of two surfaces be isometric it is necessary that it be geodesic and that the total curvature at corresponding points be equal; it is also known that neither of these conditions is alone sufficient. This paper shows that these two conditions taken together are sufficient, except when the two surfaces are of constant total curvature.

8. Professor C. L. E. Moore: *Deformation of surfaces in 4-space.*

The author derives characteristic equations which must be satisfied if two surfaces correspond by orthogonality of elements, and shows the connection with infinitesimal deformation.

9. Professor Joseph Lipka: *On conformal parallelism.*

We consider two curved spaces V_n and V_n', one of which is a conformal representation of the other. Two linear elements in V_n are conformally parallel when their conformal representations in V_n' are parallel in the sense of Levi-Civita. The author proves a number of theorems on conformal parallelism in V_n analogous to the theorems on parallelism in V_n'. A natural family of curves in V_n plays the rôle of the geodesics in V_n'. In particular, a study is made of the motion of a linear element by conformal parallelism around an infinitesimal cycle, and of the motion of a linear element both by parallelism and conformal parallelism along a curve.

10. Professor Joseph Lipka: *On irreversible systems in dynamics.*

An irreversible field of force is defined as one in which the Lagrangian or kinetic potential has the form

$$L = L_2 + L_1 + L_0 = \tfrac{1}{2}\sum_{ik} a_{ik}\frac{dx_i}{dt}\frac{dx_k}{dt} + \sum_i \phi_i \frac{dx_i}{dt} + U,$$

where t is the time, and a_{ik}, ϕ_i, and U are functions of the coordinates only. In this paper, the author proves the theorem that if in an irreversible field of force with a given constant of energy, particles are projected from all points of an arbitrary surface in a direction such that the vector sum of the velocity and the vector ϕ_i is normal to the surface, points which they reach with equal actions (action $= \int (2L_2 + L_1)dt$) lie on a surface also cutting the paths so that the vector sum of the velocity and the vector ϕ_i is normal to the surface. (If the vector ϕ_i vanishes, there is a reversible field, and the theorem reduces to the well known theorem of Thomson and Tait.) By means of Hamilton's canonical equations and the theory of contact transformations, this theorem and its converse are proved directly for curved spaces of any dimensionality.

11. Dr. J. S. Taylor: *A 4-space representation of complex plane analytics.*

Kwietniewski and K. Kommerell have set up a one-to-one correspondence between a certain family of surfaces in ordinary 4-space S_4 and the curves (real and imaginary) of plane analytics. In the present paper the author investigates the problem of representing plane analytics as a whole in a modified 4-space Σ_4, extending for use in the 4-space the complex measures of distance and angle of complex plane analytics. The "rigid motions" of S_4 and of Σ_4 are discussed, both in relation to each other and in relation to the transformations effected on equations defining regular functions of a complex variable. Among other subgroups, the group of rotations in S_4 is obtained under which all surfaces representing regular functions remain surfaces representing regular functions. Incidentally, a family of 4-space rotations is given that has apparently been overlooked. The author makes some applications to the theory of functions of a complex variable.

12. Dr. Norbert Wiener: *Differential-space.*

The author shows that procedure to the limit in questions relating to the Brownian movement gives rise to a space of infinitely many dimensions of a character fundamentally distinct from the familiar function-space discussed by such writers as Fréchet, Volterra, and Lévy, and employs the methods of Lévy in developing integration in this space.

He discusses the introduction of orthogonal coordinates in differential-space, and develops a complete set of normal and orthogonal functionals, related to the Hermite polynomials.

13. Dr. Norbert Wiener: *Note on the series* $\Sigma(\pm 1/n)$.

The author shows that with an arbitrary haphazard selection of signs for the terms of $\Sigma(\pm 1/n)$, it is of zero probability that it diverge.

14. Professor J. L. Coolidge: *On the existence of curves with assigned singularities.*

This paper appeared in the December BULLETIN.

15. Professor J. L. Coolidge: *The Gaussian law of error for any number of variables.*

In this paper the Gaussian law of error is developed for any number of variables, by a natural extension of the classical method, and the coefficients are determined in terms of individual mean errors and correlation coefficients.

16. Mr. H. W. Brinkmann: *On conformal Riemann spaces.*

The author proves the following theorem: An n-dimensional Riemann space conformal to euclidean space can be considered as the intersection of a null hypersphere and some other hypersurface in $(n+2)$-dimensional euclidean space. By means of this theorem Schouten's necessary and sufficient condition that a Riemann space be conform-euclidean is readily derived. The method of proof is entirely elementary and can be extended to prove a similar theorem for any two conformal Riemann spaces. Theorems concerning Riemann spaces containing certain sets of n-fold orthogonal systems of hypersurfaces are also obtained by this method.

17. Professor E. S. Allen: *On certain power series with positive coefficients.*

Each coefficient of a series of the type considered is a function of all preceding coefficients. Information is obtained concerning the interval of convergence, in some cases through the expression of the coefficients in closed form.

18. Dr. Philip Franklin: *A qualitative definition of the trigonometric and hyperbolic functions.*

This paper appeared in the February BULLETIN.

19. Dr. Philip Franklin: *Tensors of given types in Riemann space.*

This paper begins by defining an arithmetic tensor in euclidean space as a tensor whose components are constant, independent of both the position of the point in question and the cartesian coordinates used, and proving some of its properties. These properties are then extended to tensors in Riemann space, the final theorem being: The most general tensor of given even order involving the g_{ij}'s and their first and second derivatives, these last linearly, is a linear combination of tensors which may be formed from the curvature tensor R^e_{fgh} and the fundamental quadratic tensor g_{ij} by multiplication and contraction. There are no such tensors of odd order.

These theorems enable us to set up many of the equations used in physics by taking the most general tensor of the type wanted, and after determining the constants by special considerations equating it to zero. The paper has been accepted for publication by the PHILOSOPHICAL MAGAZINE.

20. Mr. B. Z. Linfield: *A contribution to particle geometry.*

In this paper the varieties of a universe of particles are classified according to the nature of the subvariety which is embedded in a variety and which is in the neighborhood of an arbitrary element (particle) of this same variety. After considering the n-dimensional, normal varieties of the second order, S_n^2, the author concludes with the following theorem: There exists a regular map of regions whose boundaries are convex which is equivalent to an arbitrary regular map.

21. Dr. H. E. Bray: *Proof of a formula for an area.*

This paper will appear in full in this BULLETIN.

22. Professor C. F. Craig: *On the Riemann zeta function.*

In this note a functional relation for the Riemann zeta function is established and its relation to the well known formula given by Riemann is pointed out. It is believed that the relation is new.

23. Professor Florian Cajori: *Mathematical signs of equality.*

Recorde's sign = of 1557 did not again appear in print until 1618, but after 1631 it gained a firm foothold in England. Certain Continental writers used = in the seventeenth century in four different senses. Seven different signs of equality were proposed, but Descartes' ∞ was the only serious rival, which about 1680 threatened to displace = on the Continent. The two inventors of the calculus, Leibniz and Newton, favored = and apparently brought about its final victory.

24. Professor E. T. Bell: *Extended class number relations.*

A sum of the form $\Sigma X(n - \sigma^2)$ in which X is any class number function and Σ extends to certain of the integers σ

rendering $n - \sigma^2 > 0$ is here called principal of order c and degree μ if each σ/c is the μth power of an integer. A relation involving a principal sum of order c and degree μ is called extended of rank λ and range (A, B), where $A < B$, if the relation involves also the arbitrary even function f having as arguments only integers whose λth powers lie between A and B. By a simple new method a closely interconnected set of 29 such extended relations is found in which c is an arbitrary constant integer $\neq 0$, μ is an arbitrary odd constant integer > 0, λ is an arbitrary even constant integer > 0, and A, B $(A < B)$ are arbitrary constants. Hitherto only instances of the cases $c = 1, 4$, or a prime, $\mu = 1$, $\lambda = 2$, $A = 0$, $B = \infty$, $f(x) = 1$ for all values of x have appeared. For these special values the extensions degenerate to known relations of Kronecker, Liouville, and Humbert, or to others of similar types.

25. Professor R. L. Moore: *Concerning the uniform convergence of certain sequences of equicontinuous arcs.*

Suppose that (1) AB and CD are two parallel lines, (2) for every n, A_n and C_n are points lying on AB and CD respectively and A_nC_n is a simple continuous arc which, except for its end points, lies between AB and CD, (3) the sequence of arcs $A_1B_1, A_2B_2, \cdots$ is equicontinuous and has, as its limiting set, an arc AC which lies, except for its end points, entirely between AB and CD. Then there exists, for each n, a one-to-one continuous correspondence, T_n, between AC and A_nC_n such that the set of correspondences $T_1, T_2, T_3, \cdots$ is equicontinuous and such that, *with respect to this set of correspondences, the sequence A_1C_1, A_2C_2 converges uniformly to AC*, that is to say, it is true that for every positive number ϵ there exists a positive number δ_ϵ such that if $n > \delta_\epsilon$ then, for every point P of AC, the distance from P to $T_n(P)$ is less than ϵ.

It is shown that a sequence of arcs $A_1C_1, A_2C_2, \cdots$ is not necessarily equicontinuous* even though there exist two equicontinuous sequences of functions $f_1(t), f_2(t), \cdots$ and $F_1(t), F_2(t), \cdots$ such that, for every n, A_nC_n is represented parametrically by the pair of equations $x = f_n(t)$ and $y = F_n(t)$ $(0 \leqq t \leqq 1)$.

26. Professor R. W. Burgess: *A non-linear partial regression equation.*

In the theory of partial regression equations, it is assumed that each variable is a linear function of the others. This assumption implies, in the case of three variables, that the coefficient of regression of one variable on a second is constant no matter what values are assigned to the third variable. As in certain statistical problems this seems contrary to the essential nature of the relationship between the variables, the author suggests the use of the equation $x_1 = ax_2x_3 + bx_2 + cx_3 + d$, and has determined the best values of the coefficients a, b, c, d, in the sense of the method of least squares. He has applied his method to a further analysis of H. L. Moore's problem of predicting the yield of cotton from the May rainfall and June temperature, and finds that the new method results in an improvement in accuracy of prediction.

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*Cf. TRANSACTIONS OF THIS SOCIETY, vol. 22 (1921), p. 42.

27. Professor B. H. Camp: *On a short method of least squares.*

It is known that, when the independent variables of a set of observational equations are equi-spaced, certain simplifications are available which greatly shorten the least square solution (cf. H. M. Roeser, SCIENTIFIC PAPERS OF THE BUREAU OF STANDARDS, No. 388). It is shown by the author that these may frequently be employed, with negligible error, when the independent variables are only approximately equi-spaced. Expressions for the error are derived which may be computed quickly, after the short solution has been obtained, and which also permit the computer to estimate roughly, in advance of his computation, the size of the error to be expected. This paper will appear in the ANNALS OF MATHEMATICS.

28. Professor Olive C. Hazlett: *Finiteness theorems for formal modular covariants.*

By the aid of a symbolic notation explained in a previous paper, the author proves that all formal modular covariants of a system of binary forms with respect to the Galois field $GF[p^n]$, of order p^n, are congruent, in the field, to polynomials in a finite number of such covariants.

29. Professor R. W. Marriott: *Determination of all types of subgroups of the group of order p^4, where p is a prime.*

The exact number of every type of subgroup of order p^α, $\alpha < 4$, has been determined, in this paper, for every group of order p^4, and these subgroups expressed in terms of the generating operations of the group of order p^4. The following theorem was made use of: Let Gp^m be a group of order p^m of a certain given type, and Gp^α a group of order p^α of specified type, $\alpha < m$. Let N be the number of ways operations can be selected from Gp^m which satisfy the defining relations of Gp^α, but no relations independent of these. Then the number of subgroups of Gp^m of order p^α and type Gp^α is equal to N divided by the order of the group of isomorphisms of Gp^α.

The author takes for generating elements of Gp^a the most general operations in the group of order p^m, subject these operations to the defining relations of Gp^a, and impresses on them the further condition that they shall obey no relations other than these defining relations of Gp^a. The method leads always to a set of congruences involving the exponents of the generators, the number of solutions of which gives the value of N.

30. Professor O. E. Glenn: *The sieve of Eratosthenes.*

The distribution of primes in the infinite sequence of positive integers is the subject of many researches dating from the work of Eratosthenes (276 B. C.). Tables of figurate numbers defined in the author's paper presented at the Chicago meeting, April, 1922, contain primes distributed practically at random. The method there explained for finding the figurate numbers which are divisible by a given prime p is equivalent to a generalization to two dimensions of Eratosthenes' sieve. We prove that for extensive classes of plexuses the complete description of divisibility by p may be obtained from a finite period parallelogram of residues recurrent to infinity within the plexus, and the method of the sieve gives general results. The developments of the paper connect with some ancient problems in numbers and the finished manuscript is to present extensive lists of primes from typical figurate number tables.

31. Professor G. A. Miller: *New applications of a fundamental theorem of substitution groups.*

The fundamental theorem considered in this paper may be stated as follows: If the cycle $a_1a_2\cdots a_l$ appears in at least one substitution of the group G and if we count this cycle and its conjugates under G as many times as each of them appears in different substitutions of G, then the total number of letters in these cycles is equal to the order of G. From this theorem one can readily obtain the following useful theorem: The number of the different sets of conjugate cycles under any substitution group of degree n is $n - k$, k being the number of its systems of intransitivity. In particular, all the possible transitive groups of degree n have in common the property that each of them contains the same number of sets of conjugate cycles. The author aims to exhibit the applications of these theorems by using them in a new determination of the substitution groups of degrees four and five, and in finding exponents to which numbers belong with respect to a modulus which is a power of a prime number.

32. Professor G. A. Miller: *New proof of the theorem that every prime power group contains invariant operators.*

The author shows that his proof of Sylow's theorem * can readily be extended to include a proof of the fundamental theorem that every group of order p^m (p a prime) contains at least one invariant operator of order p. To prove this extension it is convenient to let p^β represent the highest power of p which divides n instead of the highest power of p which does not exceed n, since in both cases the number of substitutions of degree p^β and order p in the symmetric group of degree n is prime to p. The regular group G whose order is divisible by p^β but not by $p^{\beta+1}$ contains a subgroup of order p^β whose constituents are regular, and the substitutions of the symmetric group of degree n which are of degree p^β and commutative with each substitution of this subgroup of order p^β must be found in the conjoints of these regular constituents. As the number of these conjoints is prime to p, each of them contains a number of substitutions of order p which is prime to p, and the subgroup of order p^β contains a number of substitutions of order p which is prime to p, and therefore an invariant operator of order p. It follows also the number of operators of order p in a group whose order is divisible by p is prime to p.

33. Dr. C. C. MacDuffee: *Covariants and transformable systems.*

This paper appeared in the January BULLETIN.

34. Dr. C. C. MacDuffee: *A theorem on covariants of forms with an application to linear algebras.*

It is proved that for every value of j the parameters α_{ij} of a linear homogeneous transformation replacing the system of forms f_k by the system f_k' are transformed by an arbitrary transformation cogrediently with the variables x_i. This theorem gives a means of calculating the rational integral covariant of the linear algebra in n units (one of which is a principal unit) which vanishes identically when and only when the right-hand (left-hand) rank of the algebra is less than n.

35. Professor C. N. Moore: *On the Cesàro summability of the double Fourier's series.*

The main result of this paper is the theorem that the double Fourier's series corresponding to any function $f(x, y)$ that has a Lebesgue integral in the region $(-\pi \leqq x \leqq \pi, -\pi$

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* ANNALS OF MATHEMATICS, (2), vol. 16 (1915), p.169.

$\leqq y \leqq \pi$) will be summable (C, ρ, δ), where ρ and δ are any two positive constants, throughout the above region, except for a set of points of measure zero. This is the extension to double Fourier's series of a theorem with regard to ordinary Fourier's series due to G. H. Hardy.

36. Professor L. L. Silverman: *A certain type of regular transformation.*

In the study of regular transformations of the form $y_n = \sum_{k=1}^{n} a_{nk} x_k$, several authors have defined a_{nn} by means of a function $f(x)$ satisfying certain conditions in the interval $(0, 1)$, and have then defined the numbers a_{nk}, $k < n$, by the condition that the given transformation be permutable with the special transformation $y = (1/n)\sum_{k=1}^{n} x_k$. These transformations, which we shall call *of type A*, include the definitions of Cesàro and Hölder for special choice of $f(x)$. The present author studies the transformations *of type B*, for which $a_{nn} = 1 - f((n-1)/n)$, $a_{nk} = f(k/n) - f((k-1)/n)$, $k < n$, where $f(x)$ satisfies the same conditions in the interval $(0, 1)$ as in the case of transformations of type A. An objection to this new type of transformation is that for no choice of $f(x)$ does it represent any of the transformations of Cesàro or Hölder, except in the case of orders 0 and 1. On the other hand, if $f_r(x)$ corresponds to C_r, the transformation of Cesàro of order r, according to the plan of type A, then the transformation B_r, obtained from $f_r(x)$ according to the plan of type B, is proved to be more general than C_r, for each positive integer r greater than unity. The transformations of type B may be written in the form of an integral, which is identical with the formula for the transformations of functions of a continuous variable.

37. Professor H. B. Phillips: *Nets and the Dirichlet problem.*

The author considers the function defined on a rectangular net by the condition that its value at each node is equal to the average of its values at the adjacent nodes. By considering nets with progressively finer mesh he approximates to the solution of the Dirichlet problem in the plane.

38. Professor H. C. M. Morse: *When does the inverse problem of the calculus of variations lead to an integral of arc length?*

The general inverse problem of the calculus of variations is to find an integral for which a given two-parameter family of curves are the extremals in case the integral be minimized. Darboux has shown that the general problem has an infinite number of solutions. The present writer inquires what are the conditions on the given family of curves that they may serve as extremals arising from an integral of arc length.

39. Mr. Louis Weisner: *Groups whose maximal cyclic subgroups are independent.*

A cyclic subgroup of a group is said to be a maximal cyclic subgroup of the group if it is not contained in any larger cyclic subgroup of the group. The present paper, which is the author's dissertation, concerns itself with groups whose maximal subgroups are independent; that is, groups in which no two maximal cyclic subgroups have an element in common besides identity. The cyclic group and the type $(1, 1, 1, \cdots)$ abelian groups are the only abelian groups having this property. Of prime power groups, the only ones are the cyclic, dihedral, the type $(1, 1, 1, \cdots)$ abelian group, and groups conformal with the type $(1, 1, 1, \cdots)$ abelian group. The only other solvable groups having the property are those which contain an invariant cyclic subgroup, and the octahedral group, which falls into a class by itself. All other such groups are insolvable and usually simple. General properties of such groups are also considered in the paper.

40. Dr. George Rutledge: *The polynomial determined by* $2n+1$ *evenly spaced points.*

Examples of the polynomials whose general form the author discusses are the following: $(4 - 3x^2)/4$, $(52 - 51x^2 + 36x^4)/52$, $(3{,}472 - 3{,}468x^2 + 3{,}321x^4 - 2{,}187x^6)/3{,}472$, $(48{,}412 - 48{,}409x^2 + 48{,}204x^4 - 44{,}928x^6 + 27{,}648x^8)/48{,}412$, which agree in value with the function $1/(1+x^2) = 1 - x^2 + x^4 - x^6 + \cdots$ for (respectively) three, five, seven, and nine equally spaced values of x on the interval $x = -1/\sqrt{3}$ to $x = +1/\sqrt{3}$, end values included. These polynomials are the basis of many well known interpolation formulas.

41. Dr. E. L. Post: *Visual intuition in spherical and elliptic space: Einstein's finite universe.*

The method of the present paper is almost the same as the one used in the author's paper on Lobachevskian intuition. However, the difficulties involved in the conception of the finiteness of both the spherical and elliptic spaces and the one-sidedness of the latter require further elucidation. This is accomplished by imagining a plenitude of the straight lines of these spaces, uniformly distributed in them, and noting just how they bind the spaces together; or by noting how the spaces can be broken up into a finite number of adjacent but nonoverlapping cells. In both cases the desired intuition is obtained by noting how these straight lines and cells vary with changes in the position of the observer. Since Einstein's finite universe is one of these types, the present paper may serve to dispel some popular misconceptions.

42. Dr. E. L. Post: *A non-Weierstrassian method of analytic prolongation.*

The present method arose from an attempt to find satisfactory necessary and sufficient conditions that a function of a real variable be an analytic function. It was found that this was so when and only when a certain straight line construction which makes a network of broken lines correspond to a subdivision of a certain portion of the plane into squares gives in the limit a network of curves through whose parameterization a transformation of the plane is defined whose corresponding functions have continuous total differentials. Since at the same time the method gives the values of the original function over a region of the complex plane, whereas originally it was known only along a segment of the real axis, it appears as a new method of analytic prolongation. By repeated application of the method, the function can be explored throughout its domain of existence. The method is that used by R. G. D. Richardson in 1917. In its particular development it yields a neat formula for the coordinates of an arbitrary point of intersection of the network as a finite series whose terms approach those of a Taylor's expansion.

43. Dr. E. L. Post: *A new method for generalizing e^x in the complex domain.*

The method in question is simply the one presented in the preceding paper. In the present case, the broken line network in the plane of the dependent variable becomes a set of lines radiating from the origin crossed orthogonally by a spider's web of broken lines. The lengths cut off along these radiating lines are found to be proportional to their distances from the origin, a fact which easily enables us to find that, in the limit, the network of broken lines is replaced by a pencil of lines crossed orthogonally by a series of concentric circles.

44. Dr. E. L. Post: *A simple geometric proof of the equality of the Brochardt angles of a triangle.*

It is easy to obtain a simple trigonometric proof of the equality of the Brochardt angles of a triangle, but as far as the author is aware no simple geometric proof of this fact is extant. The present note supplies such a proof.

45. Professor I. J. Schwatt: *Method for the summation of series in which groups of positive terms alternate with groups of negative terms.*

The author considers certain series in which the groups of positive terms and the groups of negative terms alternate and contain m terms each. He represents these series as a sum of a finite number of integrals I_k, and he has obtained a formula which enables him to find the value of I_k and similar integrals without going through the cumbersome work of integration.

46. Mr. Hillel Poritsky: *On curves kinematically related to a given curve.*

This paper appeared in the February BULLETIN.

47. Professor F. L. Hitchcock: *A method for the numerical solution of integral equations.*

The purpose of this paper is to obtain a method requiring as little as possible in the way of evaluation of definite integrals. The Fredholm equation is treated over the interval from 0 to 1. It is shown that the only necessary definite integrals are (1) the integral of the given function $s(x)$ over this interval or over any desired finite part of this interval, and (2) the integral of the kernel $K(x, y)$ over a unit square or over any desired finite square contained within the unit square. The solution is obtained by a definite rule and the computation is suitable for carrying out with a calculating machine.

48. Professor Elizabeth B. Cowley: *Further generalizations of the old mathematical puzzle of three vessels with capacities of* 8, 5 *and* 3.

This is a further development along the lines of the paper presented at the October meeting. Here the problem of four vessels is studied, starting with the cases where $A = B + C + D$. It is found that, although certain cases can be solved by repeated use of one or both of the two methods employed for the three-vessel problem, it is necessary to devise other methods to meet the situation in cases where the three-vessel methods are not applicable. Some of these new methods can also be used in cases where the three-vessel methods are applicable. The conditions for the success or failure of the various methods are found. Certain modifications are necessary in cases where $A \gtrless B + C + D$.

49. Mr. F. H. Murray: *Real solutions of certain linear differential equations of the second order.*

This paper is given to a study of the real bounded solutions of certain linear differential equations of a type discussed by Poincaré, Bohl, and Wiman. Besides a synthesis of some of the results obtained by these authors, certain new results are obtained by methods of successive approximations. In particular, an analytic expression for the general integral is obtained, from which many of the characteristic properties of the solutions follow immediately. A generalization of some of the results obtained by Bohl for linear equations with quasi-periodic coefficients is also given.

50. Mr. F. H. Murray: *A type of arc-describing instrument.*

In this paper is described a type of instrument* for drawing arcs of circles of very large radius.

51. Mr. F. H. Murray: *Certain isosceles triangle solutions in the problem of three bodies.*

The aim of this paper is to deduce a certain inequality, satisfied by the isosceles triangle solutions with axis of symmetry, and non-coplanar, in the problem of three bodies, for large negative values of the energy constant. From this inequality and another found by J. Chazy, one can show that the solutions under consideration possess "complete stability," in the sense of Poincaré (*Les Méthodes Nouvelles*, III).

52. Professor J. W. Alexander: *Invariant points of surface transformations.*

Let $A_1, A_2, \cdots, A_p, B_1, B_2, \cdots, B_p$ be a canonical system of cuts on a closed surface of genus p, and let Δ be a transformation of the surface into itself carrying the cuts A_i and B_i into the cuts $\overline{A}_i$ and $\overline{B}_i$ respectively. Then

$$\sum_j \tau_j = 1 + \tau + \sum_i \{N(B_i\overline{A}_i) + N(\overline{B}_iA_i)\},$$

where $\Sigma\tau_j$ denotes the sum of the indices of the fixed points of the transformation, τ the index of the transformation itself, and $N(AB)$ the intersection number of the curves A and B.

53. Mr. G. E. Raynor: *On Dirichlet's problem for irregular boundaries.*

The author gives an elementary solution of Dirichlet's problem under less restricted boundary conditions than those generally assumed.

R. G. D. Richardson,
Secretary of the Society.

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* These instruments are fully described in Letters Patent of the United States, No. 1,427,493, dated Aug. 29, 1922.

THE EVANSTON MEETING OF THE SOCIETY

The fiftieth regular meeting of the Chicago Section of the American Mathematical Society, being the eighteenth regular western meeting of the Society, was held at Northwestern University, Evanston, Illinois, on Friday, December 29, 1922. The attendance at this meeting was about forty-five, among whom were the following thirty-seven members of the Society:

Ballantine, W. S. Beckwith, A. D. Campbell, Chittenden, Coble, Curtiss, H. T. Davis, Dickson, Dowling, Dresden, Edington, Gokhale, Gouwens, Lane, Logsdon, McCain, McClenon, MacMillan, March, Marshall, T. E. Mason, C. N. Mills, E. J. Moulton, Plapp, Schottenfels, Skinner, J. H. Taylor, Theobald, E. L. Thompson, B. M. Turner, W. J. Wagner, R. L. Wilder, K. P. Williams, R. E. Wilson, F. E. Wood, Yanney, Zehring.

At the business meeting on Friday afternoon a resolution, proposed by Professor Skinner, expressing the Society's appreciation of the reception accorded them at Northwestern University was unanimously adopted. On Friday evening a dinner was held at the North Shore Hotel at which twenty-five persons were present. Toasts were responded to by Professors Holgate, Dickson, Curtiss, and Dresden.

At the sessions on Friday Professor Coble, Chairman of the Section, presided, relieved by Professor Curtiss in the forenoon and by Professor Dickson in the afternoon. Titles and abstracts of the papers read at this meeting are given below. Professor Moore's paper was presented by Mr. Wilder. The papers of Professor Jackson and Dr. Camp, and Professor Chittenden's first paper were read by title.

1. Professor E. P. Lane: *Ruled surfaces with generators in one-to-one correspondence.*

The author bases a projective theory of ruled surfaces, whose generators are in one-to-one correspondence and skew to each other, on a system of four ordinary linear first order differential equations in four dependent variables. This system is reduced to a canonical form by referring the surfaces to their intersector curves. The intersector curves, which are analogous to asymptotics, and certain other curves analogous to flecnode curves, are studied.

The general theory is applied in the investigation of Green-reciprocal ruled surfaces. Several of their properties are developed. In particular should be mentioned a characterization of the directrix congruences in terms of simple geometrical concepts.

2. Mr. R. L. Wilder: *Some theorems on continuous curves, with special reference to continuous curves that contain no simple closed curve.*

Among the theorems established are: (I) A sufficient[*] condition that a continuum M should be a continuous curve is that every two points that lie in a connected subset of an open subset of M lie in a sub-continuum of that open subset; (II) If N is a closed proper subset of a continuous curve M, then $M - N$ is a countable set of domains with respect to M, no two of which have a point in common, and if M contains no simple closed curve, then for $\epsilon > 0$, at most a finite number of these domains is of diameter $> \epsilon$; (III) A continuous curve M that contains no simple closed curve consists of (1) a sequence of arcs $C_1, C_2, C_3, \cdots$ every two of which have in common at most a point which is not interior to both, and such that (i) n being any positive integer, $C_1 + C_2 + \cdots + C_n$ is a continuous curve M_n and a proper subset of M, (ii) for $\epsilon > 0$ there exists a number ρ such that for $n > \rho$, $\delta(C_n) < \epsilon$ (δ meaning "diameter of") and $\delta(d) < \epsilon$, where d is any one of the maximal domains with respect to M lying in $M - M_n$; and (2) a totally disconnected set of non-cut points, P_ω, each of which is a limit point of the sequence $M_1, M_2, M_3, \cdots$, and containing all the non-cut points of M.

3. Mr. R. L. Wilder: *An analysis of the point set which constitutes the boundary of a complementary domain of a continuous curve.*

It is shown that the boundary β of a complementary domain of a continuous curve is the sum of, at most, a countable set of arcs and simple closed curves together with a totally disconnected set of limit points. It is also shown that: (I) Every closed and connected subset of β is itself a continuous curve, (II) β cannot contain, for any positive number ϵ, an infinite set of arcs of diameter $> \epsilon$, such that every two of these arcs have in common at most a point which is not an interior point of both, (III) If N is any connected subset of β, then every two points of N are the extremities of an arc lying wholly in N, (IV) A necessary and sufficient condition that the boundary of a simply connected domain be a continuous curve is that every connected subset of it should be connected in the strong sense.

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[*] R. L. Moore has proved this condition necessary. See this BULLETIN, vol. 28 (1922), p. 91.

4. Professor R. L. Moore: *An uncountable non-dense closed point set each of whose complementary intervals abuts on another one at each of its ends.*

This paper appeared in full in the February number of this BULLETIN.

5. Dr. Mayme I. Logsdon: *Closed sets of rational points on a plane cubic curve of genus one.*

If the elliptic argument of a rational point, A, on a plane cubic curve of genus one is commensurable with a period of the functions of the parameter which enter in the coordinate representation of the point, the process of using A to find other rational points by the method of tangentials and secants will terminate after a finite number of such points have been found. The author has studied certain configurations which may arise, in particular when the anharmonic ratio of the cubic is rational.

6. Mr. H. T. Davis: *Report on a boundary value problem of fourth order.*

The problem studied is that of developing an existence theorem for the characteristic numbers λ_i of the system $L(u) + \lambda k(x)u = 0$, $U_i(u) = 0$ $(i = 1, \cdots, 4)$, where $L(u)$ is the most general self-adjoint expression of fourth order, the range of x is between two singular points and $U_i(u)$ are self-adjoint linear expressions in u, u', u'', u''', two of them being taken for $x = a$ and two for $x = b$. Real transformations of independent and dependent variables reduce the problem to a normal form, preserving the self-adjointness of the system. A criterion from the theory of integral equations is then applied and an existence theorem for the λ_i developed. The problem is shown to be the analogue of Sturm's boundary value problem for a system of second order. While the solution of the latter problem may be shown to depend upon the oscillation of a certain solution of an equation of second order, the solution in the present case depends upon the oscillation of a certain solution of a fifth order equation due to Halphen.

7. Professor A. B. Coble: *The extension of the Weddle and Kummer surfaces to hyperelliptic* 3*-ways of genus* 3.

The author has previously shown that the generalized Kummer surface for $p = 3$, an M_3^{24} in S_7, can be defined as the double manifold of a certain quartic locus in S_7 whose equation has coefficients which are rational in the coordinates of an Aronhold set of 7 double tangents of the normal curve of genus 3—a ternary quartic. This definition fails completely for the hyperelliptic case since then the normal curve is a ternary quintic with a triple point. To cover this case the Kummer surface is obtained as a birational transform of a generalized Weddle surface—a 3-way in S_5—which is defined by 8 points in S_5 as the ordinary Weddle is defined by 6 points in S_3, i.e., as the locus of fixed points of a Cremona involution attached to the set of points.

8. Professor A. B. Coble: *Associated sets of points.*

A set of n points in S_k, P_n^k, is *associated* with a set Q_n^{n-k-2} if complementary minors of the matrices of the coordinates of the two sets are proportional. The author considers various methods of mapping the one set upon its associated set or of projecting the one upon the other. Properties of the set in the higher dimension are thus deduced from known properties of the associated set.

9. Professor L. E. Dickson: *The rational linear algebras of maximum and minimum ranks.*

The investigation relates to linear associate algebras with a principal unit, the coordinates of whose numbers range over an arbitrary field F. Such an algebra in n units of rank n is irreducible with respect to F if and only if $f(x)$ is a power of a polynomial irreducible in F, where $f(x) = 0$ is the rank equation. An algebra over F, of rank 2, has units $1, e_1, \cdots, e_m$ such that $e_i^2 = c_i$, $e_ie_j = - e_je_i$ $(i \neq j)$, where c_i belongs to F. If at least two c_i are not zero, then $m = 3$ and the algebra is the rational generalization of quaternions. If $c_i = 0$ $(i < m)$, $c_m \neq 0$, the algebra is equivalent either to

$$e_m^2 = 1, \qquad e_me_i = e_i, (i \leqq k), \qquad e_me_j = - e_j, (k < j < m),$$
$$e_re_s = 0, (r < m, s < m,)$$

where $2k \geqq m - 1$, or to

$e_m e_{2i-1} = e_{2i}$, $e_m e_{2i} = c_m e_{2i-1}$, $(i \leqq r)$, $e_j e_k = 0$, $(j < m, k < m)$, with $m = 2r + 1$, and c_m not a square in F. Finally, if every $c_i = 0$, the square of every number of the algebra is zero (a case of the outstanding problem of nilpotent algebras), and the rather numerous algebras in fewer than eight units are found. These results are applied to the determination of all algebras over any field F in 2, 3 or 4 units.

10. Professor L. E. Dickson: *A new simple theory of hypercomplex integers.*

The definition of a system of hypercomplex integers due to A. Hurwitz and applied to all classic algebras in 2, 3 and 4 units by Du Pasquier postulates rational coordinates, a finite arithmetical basis, closure under addition, subtraction and multiplication, the presence of the n basal units $e_0 = 1, \cdots, e_n$ (or only of e_0), and that the system is a maximal. Unfortunately, no maximal system exists for the majority of algebras. If we employ any non-maximal system, it usually happens that factorization into primes is not unique and cannot be made unique by the introduction of ideals of any kind. These essential difficulties all disappear if we replace the postulate of a finite basis by the assumption that, for every number of the system of integers, the coefficients of the rank equation are all rational integers.

11. Professor Arnold Dresden: *Symmetric forms in n variables.*

This paper considers the problem of expressing a symmetric form in n variables of weight S, of k parts and of given weight of each part, as a polynomial in the elementary symmetric functions of these variables. By means of a dominance relation between partitions of S, it is possible to specify which products of elementary symmetric functions will be involved. Various theorems are given for the determination of the coefficients with which these products will appear in the final result.

12. Professor Dunham Jackson: *A general class of problems in approximation.*

The results of recent papers by the author on the method of least mth powers are extended by the admission of a more general function of the error in place of the mth power. If y denotes the error, $|y|^m$ is replaced by a function $E(y)$ which is defined and continuous for all real values of y, becomes positively infinite as y becomes infinite in either direction, and, in most of the theorems, possesses additional properties, varying from one theorem to another. Among the topics treated are the existence and uniqueness of the approximating function, the continuity of its dependence on the function for which an approximate representation is sought (a question which was not discussed explicitly in the earlier papers mentioned), and, in the case of approximation by finite trigonometric sums, the convergence of the approximating sum as its order is indefinitely increased. Some of the results are obtained by fairly direct application of the methods previously used; others call for new considerations of some delicacy.

13. Dr. W. E. Edington: *Abstract group definitions and applications.*

Three distinct infinite systems of non-abelian groups defined by the conditions $t_1^4 = t_2^4 = (t_1t_2)^2 = (t_1t_2^3)^\alpha = 1$, $t_1^3 = t_2^6 = (t_1t_2)^2 = (t_1^2t_2^2)^\alpha = 1$, and $t_1^3 = t_2^3 = (t_1t_2)^3 = (t_1t_2^2)^\alpha = 1$, are proved to exist for all values of α and the generators as substitutions are given. The groups are of order $4\alpha^2$, $6\alpha^2$, and $3\alpha^2$, respectively, and are all solvable. The first two systems are the groups of the elliptic norm curve C_α, respectively, when the invariants g_3 and g_2 are zero. The principal subgroups and the orders of the operators are determined for the groups in each system.

14. Dr. W. E. Edington: *On an infinite system of non-abelian groups of order nm^{n-1}.*

If two non-commutative operators s_1 and s_2 fulfil the conditions

$$s_1^n = s_2^n = (s_1^\alpha s_2^\alpha)^m = 1, \qquad s_1^\alpha s_2^\alpha s_1^\beta s_2^\beta = s_1^\beta s_2^\beta s_1^\alpha s_2^\alpha,$$
$$(\alpha, \beta = 0, 1, 2, \cdots, (n-1)),$$

they generate a group of order nm^{n-1}, and such a group exists for all values of m and n. The generators as substitutions are given in the paper. The groups of this system are all solvable. Several known infinite systems are special cases of this more general system.

15. Dr. W. E. Edington: *On an infinite system of non-abelian groups of order nm^n.*

Two non-commutative operators s_1 and s_2 having the property that the transforms of their product s_1s_2 by s_1 and its powers are all distinct and commutative, and also fulfilling the conditions $s_1^n = s_2^{mn} = (s_1s_2)^m = 1$, generate a group of order nm^n, and groups exist for all values of m and n. The generators represented as substitutions also possess interesting properties.

16. Professor E. W. Chittenden: *Note on a property of abstract sets which admit a definition of distance.*

Fréchet gives the following definition of a complete set (D): a set (D) with preassigned elements of accumulation is *complete* if among all possible definitions of distance in (D) for which the elements of accumulation remain unchanged there is one with respect to which (D) admits a generalization of the theorem of Cauchy regarding the convergence of sequences; and remarks that it would be interesting to know whether there exist sets (D) which are not complete. It is shown that if a set (D) contains a set H which is in its derived set H', then (D) contains a perfect set. Therefore the rational numbers with distance defined as usual form a set (D) which is not complete.

17. Professor E. W. Chittenden: *The Schmidt linear differential forms of a limited bilinear form in infinitely many variables.*

The Hellinger theory of the linear differential forms of a quadratic form in infinitely many variables can be used to obtain a representation of the general limited bilinear form which is analogous to Schmidt's representation of the unsymmetric kernel $K(x, y)$ in terms of the fundamental functions of $\overline{K}(x, y) = \int_a^b K(x, t)K(y, t)dt$, $\underline{K}(x, y) = \int_a^b K(t, x)K(t, y)dt$.

18. Professor E. W. Chittenden: *On a form of the property of Borel-Lebesgue which is independent of the closure of derived classes.*

The closure of derived classes is extensionably attainable in terms of the concept, "limiting element of transfinite order." The property of Borel-Lebesgue and the property, "perfectly compact," of R. L. Moore, may be so generalized that the new properties are equivalent in any system (V) of Fréchet which has the property 2°, that any element of accumulation of the sum of two classes is an element of accumulation of at least one of the two classes. Furthermore, if derived classes are closed (in the usual sense), the generalized properties are equivalent, respectively, to the property of Borel-Lebesgue and the property, "perfectly compact."

19. Professor K. P. Williams: *Concerning an expansion in the restricted problem of three bodies.*

In one of the Lagrange particular solutions of the problem of three bodies, the function $y(\mu)$ defined by

$$y^5 + (3 - \mu)y^4 + (3 - 2\mu)y^3 - \mu y^2 - 2\mu y - \mu = 0$$

comes into consideration (Moulton, *Celestial Mechanics*, 1st edition, p. 197). The purpose of the present paper is to determine the radius of convergence of the series that gives y.

20. Dr. C. C. Camp: *Expansions in terms of solutions of partial differential equations.* Second paper.

The author considers the equation $\sum_{i=1}^{p} L_i(u) + \lambda u = 0$ with the boundary conditions $T_{ij}(u) = 0$, $i = 1, 2, \cdots, p$; $j = 1, 2, \cdots, n_i$, where $L_k(u)$ is a partial differential expression of order n_k in x_k similar to the ordinary differential expression in x considered by Birkhoff in his thesis, and $T_{kj}(u)$ is a partial differential expression of order $n_k - 1$ such that $T_{kj}(u) = 0$ reduces to one of Birkhoff's *regular* boundary conditions in a_k and b_k by the transformation $u = p_{i=1}^{\rho} u_i(x_i)$. This gives rise to a set of p Birkhoff systems, each involving a different parameter. By employing a convergence proof similar to that used in his first paper the author shows that a function f in p variables, similarly restricted in the region $a_i \leqq x_i \leqq b_i$, may be expanded analogously in a *multiple Birkhoff series* in terms of principal solutions of the set of systems and of the adjoint set. Such a series has the same convergence properties as the multiple Fourier expansion except at points on the boundaries. The development has applications in important physical problems such as those of potential, heat flow, and wave motion.

ARNOLD DRESDEN,
Secretary of the Chicago Section.

THE FIFTEENTH REGULAR MEETING OF THE SOUTHWESTERN SECTION

The fifteenth regular meeting of the Southwestern Section of this Society was held at the University of Kansas on Saturday, December 2, 1922. The total attendance was twenty-eight, including the following sixteen members:

Ashton, Black, Candy, Garrett, Ingold, Kendall, Lefschetz, J. V. McKelvey, U. G. Mitchell, R. L. Moore, Roever, Edwin R. Smith, G. W. Smith, Stouffer, J. S. Turner, Wheeler.

Professor Ashton occupied the chair, being relieved during the afternoon session by Professor Lefschetz. The morning session and part of the afternoon session were devoted to the reading of the papers listed below. During the afternoon session Professor R. L. Moore gave a special lecture by invitation on *Continuous curves from the view-point of analysis situs*. This paper will appear in full in a later number of this BULLETIN. It was voted to hold the next meeting of the Southwestern Section at the University of Missouri. The following program committee was elected: Professors Hedrick (chairman), McKelvey, and Stouffer (secretary).

The titles and abstracts of the papers read follow below.

1. Professor A. L. Candy: *Cyclic operations on determinants.*

This paper consists chiefly of three theorems: (I) If the rows of a given determinant Δ be added cyclically s in a set, the resulting determinant Δ_s will be equal to zero, or $s \cdot \Delta$, according as n (order of Δ) and s are commensurable, or incommensurable. (II) If the elements in each column of a determinant Δ be permuted cyclically in all possible ways, and the n^n determinants thus formed be collected into groups such that each determinant in the group shall have the same subscripts in the first row, the sum of the determinants in any one of these groups will be equal to $\pm \lambda\Delta$, or zero, according as the weight of the group (the sum of the subscripts in any first row) is, or is not, a multiple of n, where λ is 1, n, a submultiple of n, or the sum of two or more such numbers. (III) The sum of the $n!$ determinants that can be formed by making the $n!$ permutations of the columns of a circulant is equal to zero.

2. Professors Claribel Kendall and G. W. Smith: *On the conditions for associativity in linear algebra.*

In a linear algebra of n units the differences $(e_ie_j)e_k - e_i(e_je_k)$ lead to n^4 quadratic forms in the n^3 constants of multiplication. The authors establish certain relations among these forms, and in the case $n = 2$ determine eight relations among them.

3. Professor G. W. Smith: *Analytic expression for line segments.*

In this paper the author sets up an analytic expression which represents $f(x)$ in a certain interval $a < x < b$ $(a > 0)$ and only in this interval, the expression being imaginary for other values of x. By a combination of these he represents analytically such geometric figures as triangles and squares.

4. Professor W. H. Roever: *Singularity in the boundary curve of the projection of a surface.*

In the boundary curve of the projection of a surface there may be points at which the curve stops abruptly. This is illustrated in the case of the orthographic projection of a torus, for which the curve in question consists of two parallels to an ellipse. If the torus is sufficiently inclined to the plane of projection the inner parallel has four cusps, and only that portion of it which lies between a particular pair of these cusps is visible and thus the inner boundary apparently stops at these points. Several methods of determining such singularities.

5. Professor Edwin R. Smith: *The determination of hidden periodicities by difference methods.*

If $y = f(x, \alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n)$ is a function of x with n distinct simple unknown periods $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_n$, it is shown that the function satisfies a difference equation of the form $u(x) - C_1u_{n-1}(x) - C_2u_{n-2}(x) - \cdots - (-1)^nC_nu_0(x) = 0$, where $C_1, C_2, C_3, \cdots, C_n$ are symmetric functions of $\cos \alpha_1, \cos \alpha_2, \cdots, \cos \alpha_n$ and $u_0(x) = y_{x+1} - y_x$, $u_i(x) = u_{i-1}(x-1) - u_{i-1}(x+1)$, $(i = 1, 2, \cdots, n)$. The values of $C_1, C_2, C_3, \cdots, C_n$ corresponding to a given set of observations may be found by any other convenient method. The method of least squares leads to results similar to those obtained by the Lagrange-Dale method.

6. Professor S. Lefschetz: *Concerning Nöther's theorem on curves traced on non-singular surfaces.*

The theorem in question states that *all algebraic curves traced on a generic surface of order $m > 3$ in ordinary space are complete intersections.*

In his Bordin prize paper the author gave the first complete proof of this important proposition, or rather of a more general one which includes it as a special case. One of the main features of the proof consists in showing that *not* all double integrals of the first kind attached to the surfaces in question have all periods equal to zero. In this paper a notable simplification of this part of the proof is given, applicable when it is desired to prove strictly Nöther's theorem and no more. It consists essentially in building up a surface of order m which has an almost obvious integral with a non-zero period.

7. Professors E. R. Hedrick and Louis Ingold: *Remarks concerning the Michelson-Morley experiment.*

Because of prevailing differences of opinion relating to the celebrated Michelson-Morley experiment a new investigation of the mathematical theory seems to be desirable. The work confirms, by an independent calculation, certain results obtained by Righi, and this is of great importance since the accuracy of his work has been called in question (OBSERVATORY, vol. 44 (1921), pp. 340–341).

8. Professor J. S. Turner: *Extensions of the Euler and Seelhoff determinants, with examples of their use in determining whether a number is prime.*

It is first pointed out that the non-simple reduced forms of the odd Euler determinants become simple forms, i.e., forms containing no product term when multiplied by 2. Hence if a prime p has the quadratic character of a non-simple reduced form, then $2p$ has a single representation by a simple form. A list of negative determinants follows in which there are three reduced forms in each genus, such that four times each non-simple form is a simple form. If p has the quadratic character of a simple form, $4p$ has a single representation by that form. Seelhoff's list (AMERICAN JOURNAL, vols. 7–8) of determinants in which certain genera contain only simple reduced forms is extended. A list of determinants follows in which certain genera contain only ambiguous reduced forms, i.e., forms which are simple or become simple on multiplication by 2. The final list contains determinants for which all the forms in certain genera are simple or become simple on multiplication by 4. The advantages of the new determinants are explained and illustrated by examples. These examples also illustrate an improved method of exclusion.

E. B. STOUFFER, *Secretary of the Section.*

A SET OF AXIOMS FOR LINE GEOMETRY*

BY M. G. GABA

1. *Introduction.* In 1901 Pieri proposed a set of axioms for line geometry in terms of line and intersection.† That Pieri's set of eleven postulates was not independent was shown by Hedrick and Ingold in 1914; they proposed a simpler and more elegant set of but five independent axioms, using the same undefined concepts.‡ Both of these sets are for geometries equivalent to the general three-space geometry established by axioms A_1, A_2, A_3, E_0, E_1, E_2, E_3 and E_3' of Veblen and Young.§

In this paper is given a set of six independent axioms in terms of line as an undefined element and an undefined class of one-to-one correspondences among the lines called collineations. There is introduced but one defined term before the complete statement of the axioms. To make a proper projective space it has usually been necessary not only to add a postulate of projectivity but also a sequence of definitions for such concepts as perspectivity, projectivity, etc., to give that postulate content. If to our set a seventh postulate is added, we have a proper projective three-space without the intervention of any additional defined concepts.‖

2. *Postulates.* Our basis is a class of undefined elements, called *lines*; an undefined class of one-to-one correspondences, or transformations, among the lines, called *collineations*; and

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* Presented to the Society, Nov. 27, 1920.

† *Sui principi che regno la geometria delle rette*, TORINO ATTI, vol. 36 (1901), pp. 335–351.

‡ *A set of axioms for line geometry*, TRANSACTIONS OF THIS SOCIETY, vol. 15 (1914), pp. 205–214.

§ *A set of assumptions for projective geometry*, AMERICAN JOURNAL, vol. 30 (1908); *Projective Geometry*, vol. 1, Boston, 1910.

‖ Another set of postulates equivalent to the set of all seven of the axioms is the first seven given by the author in his paper *A set of postulates for general projective geometry*, TRANSACTIONS OF THIS SOCIETY, vol. 16 (1915), pp. 51–61.

a class of defined classes of lines, called *fields*. By a one-to-one correspondence among the lines we mean that to every line α there corresponds one and only one line α', and no α' is the correspondent of two distinct lines α_1 and α_2. This concept of one-to-one correspondence may be said to belong to the domain of pure logic; we shall therefore not consider it in our basis as an undefined mathematical notion.* We shall define a field as a class of at least three lines such that: (1) every collineation that makes any two distinct lines of the class correspond to two distinct lines of the class leaves the class invariant; (2) a collineation exists that will make any two distinct lines of the class correspond to any two distinct lines of the class; (3) the class has no proper super-class possessing property (2).

Our axioms, whose technical independence is shown by examples given at the end of this paper, are as follows:

I. *There exists a line.*

II. *If α is a line, there exist five distinct fields containing α.*

III. *If a collineation τ_1 exists such that τ_1 makes correspond to the set of lines A the set of lines B, and if a collineation τ_2 exists such that τ_2 makes correspond to the set of lines A the set of lines C, then a collineation τ_3 exists such that τ_3 makes correspond to the set of lines B the set of lines C.*

IV. *If no collineation exists that makes correspond to the field A the field B, and if no collineation exists that makes correspond to the field A the field C, then a collineation τ exists that makes correspond to the field B the field C.*

V. *If A and B are fields such that a collineation τ exists that makes correspond to the field A the field B, then the fields A and B have a line in common.*

VI. *If A, B and C are fields such that A and B have a line α_1 in common, and B and C have a line α_2 in common, and C and*

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* It is true, of course, that no sharp line of demarcation exists between concepts of pure logic and of mathematics; but an author may choose whether or not he will include a given concept in his own discussion. See Federico Enriques, *Problems of Science*, authorized translation by J. Royce. Open Court Publishing Co., 1914, p. 122.

A have a line α_3 *in common, then there exists a field D such that D contains the lines* α_1, α_2 *and* α_3.

VII. *If A, B, C and D are four distinct fields having a line* α *in common, and if collineations* τ_1, τ_2 *and* τ_3 *exist such that* τ_1 *makes correspond to the field A the field B,* τ_2 *makes correspond to the field B the field C, and* τ_3 *makes correspond to the field C the field D, then every collineation* τ *that leaves A, B and C invariant also leaves D invariant.*

3. *Theorems.* THEOREM 1. *If A is a field, then a collineation exists that leaves A invariant.*

Let us assume that no collineation exists that leaves A invariant. If now we let fields B and C of Postulate IV be identical with A, we are led at once to an absurdity.

THEOREM 2. *If a collineation* τ *exists that makes correspond to the field A the set of lines B (or transforms A into B, in notation,* $\tau(A) = B$*), then a collineation* τ^{-1} *exists such that* τ^{-1} *makes correspond to the set of lines B the field A:* $\tau^{-1}(B) = A$.

By hypothesis, a collineation τ exists such that $\tau(A) = B$, and, by Theorem 1, a collineation τ_1 exists such that $\tau_1(A) = A$. Hence it follows from Postulate III that a collineation τ^{-1} exists such that $\tau^{-1}(B) = A$.

THEOREM 3. *If a collineation* τ_1 *exists such that* τ_1 *transforms the field A into a set of lines B, and a collineation* τ_2 *exists such that* τ_2 *transforms the set of lines B into a set of lines C, then a* τ_3 *exists such that* τ_3 *transforms A into C.*

By Theorem 2, a collineation τ_1^{-1} exists such that $\tau_1^{-1}(B) = A$. The existence of τ_1^{-1} and τ_2, by virtue of Postulate III, implies the existence of a collineation τ_3 such that $\tau_3(A) = C$.

COROLLARY. *The resultant of a sequence of collineations on a field is a collineation.*

THEOREM 4. *Every collineation transforms a field into a field.*

Let τ be any collineation that transforms the field A into a set of lines B. Let β_1, β_2, β_1' and β_2' be any four lines of B such that β_1 and β_2 are distinct and β_1' and β_2' are distinct; and let α_1, α_2, α_1' and α_2' be the four lines of A such that $\tau(\alpha_1, \alpha_2, \alpha_1', \alpha_2') = \beta_1, \beta_2, \beta_1', \beta_2'$. Let τ_1 be any collineation such that $\tau_1(\beta_1, \beta_2) = \beta_1', \beta_2'$. From Theorem 2 and the corollary of Theorem 3, we know that a collineation $\tau_2 = \tau^{-1}\tau_1\tau$ exists such that $\tau_2(\alpha_1, \alpha_2) = \bar{\alpha}_1', \bar{\alpha}_2'$, where $\bar{\alpha}_1'$ and $\bar{\alpha}_2'$ are lines of A. But since the collineation τ_2 makes two lines of the field A correspond to two lines of A (which are distinct, since they are the correspondents of pairs of distinct lines), τ_2 leaves the field A invariant. Therefore, since $\tau^{-1}\tau_1\tau(A) = A, \tau(A) = B$, and $\tau^{-1}(B) = A$, we have $\tau^{-1}\tau_1(B) = A$, and hence $\tau_1(B) = B$.

Since A is a field, a collineation τ_3 exists such that $\tau_3(\bar{\alpha}_1, \bar{\alpha}_2) = \alpha_1', \alpha_2'$, where $\tau^{-1}(\beta_1, \beta_2) = \bar{\alpha}_1, \bar{\alpha}_2$. Therefore a collineation $\tau_4 = \tau\tau_3\tau^{-1}$ exists such that $\tau_4(\beta_1, \beta_2) = \beta_1', \beta_2'$.

We have shown that properties (1) and (2) of a field are possessed by the set of lines B. Let us now assume that B has a proper super-class B^* also possessing property (2). If $\tau'(B) = A$ and if $\tau'(B^*) = A^*$, then A^* is a proper super-class of A. Let α_1^*, α_2^* and $\alpha_1^{*\prime}$, $\alpha_2^{*\prime}$ be any two pairs of distinct lines of A^*. Furthermore let β_1^*, β_2^*, $\beta_1^{*\prime}$ and $\beta_2^{*\prime}$ be those four lines of B^* such that $\tau(\alpha_1^*, \alpha_2^*) = \beta_1^*, \beta_2^*$ and $\tau^{-1}(\beta_1^{*\prime}, \beta_2^{*\prime}) = \alpha_1^{*\prime}, \alpha_2^{*\prime}$. By our assumption, a τ_5 exists such that $\tau_5(\beta_1^*, \beta_2^*) = \beta_1^{*\prime}, \beta_2^{*\prime}$; therefore a $\tau_6 = \tau^{-1}\tau_5\tau$ exists. But $\tau_6(\alpha_1^*, \alpha_2^*) = \alpha_1^{*\prime}, \alpha_2^{*\prime}$; hence A^* possesses property (2) of a field, which is impossible if A is a field. Therefore our assumption leads to a contradiction; hence B is a field.

THEOREM 5. *If A and B are distinct fields such that a collineation exists transforming A into B, then A and B have at most one line in common.*

Let us assume that A and B have the two distinct lines γ_1 and γ_2 in common. If the collineation that makes A go over into B is τ, then let $\tau(\gamma_1, \gamma_2) = \beta_1, \beta_2$. Since γ_1, γ_2, β_1 and β_2 are pairs of distinct lines of B, a collineation τ_1 exists such that $\tau_1(\beta_1, \beta_2) = \gamma_1, \gamma_2$ and $\tau_1(B) = B$. But $\tau_1\tau(A) = A$, since γ_1 and γ_2 of A are invariant; and $\tau_1\tau(A) = B$, since $\tau(A) = B$; therefore our assumption must be wrong.

THEOREM 6. *If A is a field containing the lines α_1 and α_2, then there exists a field $\bar{A}$ also containing α_1 and α_2 and such that no collineation exists that will transform A into $\bar{A}$.*

We shall assume α_1 and α_2 distinct, since the theorem is true in the case where α_1 and α_2 are identical if it is true when α_1

and α_2 are distinct. By means of Postulate II we know that there exists a field A_1 containing the line α_1 and distinct from A. Let τ be a collineation such that $\tau(\alpha_1, \alpha_2) = \alpha_2, \alpha_1$. From Theorem 4, we have $\tau(A_1) = A_2$, where A_2 is a field containing α_2. If $A_1 = A_2$, then A_1 is $\bar{A}$, since A and A_1 have the two distinct lines α_1 and α_2 in common; hence they cannot be transformable into one another without contradicting Theorem 5. Let us now assume A_1 and A_2 distinct. Since A_1 is transformable into A_2, fields A_1 and A_2 have a line α in common. But since A and A_1 have the line α_1 in common, A_1 and A_2 the line α in common, and A_2 and A the line α_2 in common, it follows from Postulate VI that a field A' exists containing α, α_1 and α_2. If A is not transformable into A', then A' is $\bar{A}$. If A is transformable into A', then A is identical with A' by Theorem 5. In the latter case, however, A_1 is not transformable into A, since if that were possible, A and A_1 would be identical, since they have the lines α_1 and α (distinct, since otherwise $A_1 = A_2$) in common. Since α, α_1 and α_2 are distinct lines of the field A, a collineation τ_1 exists such that $\tau_1(\alpha_1, \alpha) = \alpha_1, \alpha_2$. The collineation τ_1 transforms the field A_1 into a field A_1' containing the lines α_1 and α_2. But A cannot be transformed into A_1', because, if that were possible, then, by Postulate III, A_1 would be transformable into A. Hence it follows that A_1' is $\bar{A}$ in this case.

THEOREM 7. *Every field belongs to one or to the other of two classes of fields such that collineations exist that will transform any field of one class into any field of the same class, and such that no collineations exist that will transform any field of one class into a field of the other class.*

From Theorem 6, we know that two fields A and $\bar{A}$ exist such that no collineation will transform A into $\bar{A}$. Let $[A_i]$ be the class of all fields transformable into A, and let $[\bar{A}_i]$ be the class of all fields transformable into $\bar{A}$. Let B be any field. If a collineation τ exists such that $\tau(B) = A$, then B belongs to $[A_i]$. Suppose B does not belong to $[A_i]$, then since no collineation exists transforming B into A, and no collineation exists transforming $\bar{A}$ into A, it follows from Theorem 2 and

Postulate IV that a collineation $\bar{\tau}$ exists such that $\bar{\tau}(B) = \bar{A}$; hence B belongs to $[\bar{A}_i]$. But B cannot belong to both $[A_i]$ and $[\bar{A}_i]$, for in that case it would follow from Postulate III that A could be transformed into $\bar{A}$. Any field A_i of the class $[A_i]$ will determine the class, since it is easy to show that if A_i and A_j are both members of the class and hence transformable into A, they are transformable into one another. Furthermore, no A_i of $[A_i]$ can be transformed into an $\bar{A}_j$ of $[\bar{A}_i]$, since that would make A_i belong to both classes.

We shall call the fields of one of the two classes of fields *points*, and the fields of the other class *planes*. We shall use the terminology α is a line *on* the plane p or p is *on* α to mean α is a line of p; P is a point *on* the plane p or p *on* P to mean that P and p have a line in common; P is a point *on* the line α or α *on* P to mean that α is a line of P. If two lines are on same point, they are called *copunctal;* if two lines are on the same plane, they are called *coplanar*.

THEOREM 8. *If a point P is on the line α and α is on the plane p, then P is on p.*

Since P and p have the line α in common, the theorem is obvious.

THEOREM 9. *If two lines are copunctal* (*coplanar*), *they are coplanar* (*copunctal*).

This theorem is but a restatement of Theorem 6 in the light of subsequent definitions.

THEOREM 10. *If P_1 and P_2 are distinct points* (*planes*), *then there is at least one line on both P_1 and P_2.*

See Postulate V.

THEOREM 11. *If P_1 and P_2 are distinct points* (*planes*), *there is not more than one line on both P_1 and P_2.*

See Theorem 5.

Points (planes) which are on the same line are said to be *collinear;* points (planes) which are not on the same line are said to be *non-collinear*.

THEOREM 12. *If P_1, P_2 and P_3 are three non-collinear points* (*planes*), *then there is one and only one plane* (*point*) *p such that P_1, P_2 and P_3 are on p.*

Let the line common to P_1 and P_2 be α_3, the line common to P_2 and P_3 be α_1, and the line common to P_3 and P_1 be α_2. By Postulate VI a field exists containing α_1, α_2 and α_3. If this field is a point (plane), then P_1, P_2 and P_3 coincide, contrary to the hypothesis that they are non-collinear and hence distinct. The field is therefore a plane (point). There can be but one such plane (point). Since α_1 and α_2 are distinct (otherwise P_1, P_2 and P_3 are collinear), they determine the plane (point) p. Let α_1' be a line of P_1 in p', α_2' a line of P_2 in p', and α_3' a line of P_3 in p'; hence P_1, P_2 and P_3 are on the plane (point) p'. If α_1' and α_2' coincide, then $\alpha_1' = \alpha_2' = \alpha_3$. Since α_1' and α_2' are coplanar (copunctal), they must be copunctal (coplanar). If α_1' and α_2' are distinct, they determine a point (plane) P_3'. Now P_3' cannot be on α_3 since that would make α_1' and α_2' coincide. By the part of the theorem already proved, a plane (point) p'' exists such that P_1, P_2 and P_3' are on p''. But p'' contains α_1' and α_2'; therefore p'' is p'. Hence the plane (point) p' contains α_3. In like fashion it can be shown that p' contains α_1 and α_2; hence p' is p.

COROLLARY 1. *If P_1 and P_2 are distinct points (planes), and if P_1 and P_2 are both on the plane (point) p, then the line α common to P_1 and P_2 is on P.*

COROLLARY 2. *If P is a point (plane) and α is a line such that P is not on α, then there is one and only one plane (point) determined by P and α.*

THEOREM 13. *If P_1, P_2, P_3 are three points (planes) not on the same line, and α is a line joining P_2 and P_3, the class $[P_i]$ of all points (planes) such that every point (plane) of $[P_i]$ is collinear with P_1 and some point (plane) on α, is such that every point (plane) of $[P_i]$ is on the plane (point) p determined by P and α, and every point (plane) on the plane (point) p is a point (plane) of $[P_i]$.*

Let P_i be a point (plane) collinear with P_1, and $\bar{P}_i$ a point (plane) on α. $\bar{P}_i$ is on p, since $\bar{P}_i$ is on α, by Theorem 8. Let the line determined by P_1 and $\bar{P}_i$ be α_i. Then α_i is on p by Corollary 1, Theorem 12. But P_i is on α_i; therefore P_i is on p.

Let P' be any point (plane) of p. P_1 and P' determine a line α' (unless P' is P_1, in which case P' obviously belongs to $[P_i]$). The lines α' and α are distinct since α is not on P_1, and they are copunctal (coplanar) since both lines are on p. Hence α and α' determine a point (plane) $\bar{P}'$. Hence P' is collinear with P_1, and $\bar{P}'$ a point (plane) on α; consequently P' belongs to $[P_i]$.

THEOREM 14. *If P_1, P_2, P_3 are points (planes) not all on the same line, and P_4 and P_5 ($P_4 \neq P_5$) are points (planes) such that P_2, P_3, P_4 are on a line and P_1, P_3, P_5 are on a line, there is a point (plane) P_6 such that P_1, P_2, P_6 are on a line and also P_4, P_5, P_6 are on a line.*

Since the points (planes) P_1, P_2 and P_3 are non-collinear, they determine a plane (point) p. The three lines α_1 common to P_2 and P_3, α_2 common to P_1 and P_3, and α_3 common to P_1 and P_2, are on p by Corollary 1, Theorem 12. Since P_4 is on α_1, and P_5 is on α_2, P_4 and P_5 are also on p. Hence the line α common to P_4 and P_5 is on p, and consequently α is copunctal (coplanar) with α_1. If α and α_1 are distinct, they determine a point (plane) which is P_6. If α and α_1 coincide, then any point (plane) on α will serve as a P_6.

THEOREM 15. *If α_1 and α_2 are lines, then α_1 and α_2 are on the same number of points (planes).*

Since collineations exist that transform any point (plane) into any point (plane) and any line in a point (plane) into any other in the same point (plane), a collineation τ exists that transforms α_1 into α_2. Every point (plane) on α_1 is transformed by τ into a point (plane) on α_2, and every point (plane) on α_2 comes from a point (plane) on α_1. Hence α_1 and α_2 are on the same number of points (planes).

THEOREM 16. *All points (planes) are not on the same line.*

Let us assume that every point (plane) is on a line α. Let τ be a collineation that transforms α into a distinct line α'. If we transform every point (plane) by τ, we get all the points (planes) again, since distinct fields transform into distinct fields. But all points (planes) now contain α' as well as α. Therefore it follows that there is but one point (plane) P.

Every line is in P, since, by Theorem 6, every line is in a point (plane). No other field can exist, since, if it did, it would be a proper sub-class of P, contrary to the definition of field. But if there is only one field, Postulate II is contradicted. Hence our original assumption is false and the theorem proved.

THEOREM 17. *If α is a line, then the number of planes on α is the same as the number of points on α.*

Since, by Theorem 16, all planes are not on α, there exists a line $\bar{\alpha}$ non-coplanar with α. Every point on $\bar{\alpha}$ determines, with the line α, a different plane, since if two distinct points $\bar{P}_1$ and $\bar{P}_2$ on $\bar{\alpha}$ and α were on the same plane, the lines $\bar{\alpha}$ and α would be coplanar. Hence there are at least as many planes on α as there are points on $\bar{\alpha}$. Likewise every plane on α determines, with the line $\bar{\alpha}$, a different point. Hence there are at least as many points on $\bar{\alpha}$ as there are planes on $\bar{\alpha}$. Since, by Theorem 15, the number of points on α and on $\bar{\alpha}$ are equal, and since the number of planes on α must equal the number of points on $\bar{\alpha}$, it follows that α is on the same number of points as α is on planes.

THEOREM 18. *There are at least three points (planes) on every line.*

By Postulate II, every line is on at least five fields; and, since the number of planes and points on a line is the same, the theorem follows immediately.

THEOREM 19. *All points (planes) are not on the same plane (point).*

Let p be any plane (point). Since there is more than one plane (point), there exists a line α not on p. If two points (planes) of α were on p, then α would be on p. Hence at most one of the points (planes) of α can be on p.

4. *Comparison with Ordinary Geometry.* Let us now compare our geometry with the general three-space geometry of Veblen and Young.* If we identify our lines as lines, collineations as projectivities, and fields as either points or planes in the Veblen-Young geometry, then our axioms are theorems in

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* Loc. cit.

their geometry. Incidently, the consistency of our axioms is thus established.

On the other hand, let the undefined *line* of Veblen and Young be our *line*; their undefined *point*, our *point*; and their *on* our *on*. Consider their assumptions of alignment A:

A_1. *If A and B are distinct points, there is at least one line on both A and B.*

A_2. *If A and B are distinct points, there is not more than one line on both A and B.*

A_3. *If A, B, C are points not all on the same line, and D and E $(D \neq E)$ are points such that B, C, D are on a line and C, A, E are on a line, there is a point F such that A, B, F are on a line and also D, E, F are on a line.*

We see that, except for notation, A_1 is our Theorem 10, A_2 is Theorem 11, and A_3 is Theorem 14. The *plane* defined by Veblen and Young is equivalent to our *plane*, by virtue of Theorem 13.

The assumptions of extension, substituting for the usual E_3' an equivalent axiom also given by Veblen and Young, are as follows:

E_0. *There are at least three points on every line.*

E_1. *There exists at least one line.*

E_2. *All points are not on the same line.*

E_3. *All points are not on the same plane.*

E_3'. *Any two distinct planes have a line in common.*

E_0 is our Theorem 18; E_1 is Postulate I; E_2 is Theorem 16; E_3 is Theorem 19; and E_3' is Theorem 10. We have therefore proved that our geometry is equivalent to that of Veblen and Young.

The assumption of projectivity is as follows:

P. *If a projectivity leaves each of three distinct points of a line invariant, it leaves every point of the line invariant.*

It is evident that P is a rephrasing of Postulate VII in terms of point.

5. *Independence Examples.*—1. No lines exist. Hence neither collineations nor fields exist.

—2. Lines are all the lines of a projective three-space and an extra line not in that three-space. Collineations are all the projectivities on the lines of the three-space, but in addition leave the extra line invariant. Fields are therefore the points and planes of the three-space.

—3. Lines are all the lines of a projective finite three-space. Collineations are all the projectivities on the three-space and an extra transformation τ, such that τ transforms every line α into a line α' skew to α. Fields are therefore the points and planes of the three-space.

—4. Lines are all the lines of two projective three-spaces having no line in common. Collineations are all the projectivities on the two three-spaces such that each of the three-spaces is left invariant. Fields are therefore the points and planes of the two three-spaces.

—5. Lines are all the lines of two projective three-spaces having no line in common. Collineations are all the projectivities on the two three-spaces such that each of the three-spaces is left invariant, or such that the two three-spaces are interchanged. Fields are therefore the points and planes of the two three-spaces.

—6. Lines are all the lines of a euclidean plane. Collineations are all the projectivities such that parallel lines are transformed into parallel lines. Fields are therefore pencils of lines.

—7. Lines are all the lines of an improperly projective three-space. Collineations are all the projectivities in such a three-space. Fields are therefore the points and planes of that three-space.

University of Nebraska

SHORTER NOTICES

Théorie Mathématique des Assurances. By P. J. Richard et E. Petit. Second edition, revised throughout and brought up to date. Paris, Librairie Octave Doin, Gaston Doin, Editeur, 1922. Volume I, 455 pp.; Volume II, 320 pp.

These two volumes appear in the series entitled ENCYCLOPÉDIE SCIENTIFIQUE–BIBLIOTHÈQUE DE MATHÉMATIQUES APPLIQUÉES. The first volume, book one, contains two chapters on the theory of probability, with development of elementary theorems, the theorem of Bernoulli, Baye's rule, and a consideration of insurance from the point of view of a game of chance.

Book two contains a chapter on life and mortality tables and gives definitions of the rate of mortality and derived functions. The effects of age, sex, occupation, climate, selection, and other causes on the rate of mortality are explained. Various methods of constructing and graduating the rate of mortality are described, including graphical, mechanical and analytic adjustment.

Several proposed laws of human mortality including the formulas of De Moivre, Gompertz and Makeham and other generalizations are given. The old and the modern French tables for annuitants and for insured are described together with methods of adjustment and graduation to Makeham's law.

Chapter II is divided into four parts. The first part relates to annuities-certain and life annuities and develops the general theory of various types of annuities and their expression in terms of commutation symbols. Part two deals with last survivor annuities and annuities to x after the death of y. The theory of single premium insurance benefits on one life and on joint lives is developed in part three and the last part deals with a variety of forms of insurance and gives methods of deriving the net single and annual premiums.

Chapter III consists of five parts devoted respectively to gross premiums and methods of loading, mathematical reserves, contracts of insurance and alterations, dividends in insurance, and concludes with a study of the mathematical insurance risk and the stability of insurance based on the theory of risk.

This volume concludes with a very good six page bibliography of life insurance literature.

The second volume is divided into two books. The first book is devoted to insurance on the individual and consists of five chapters.

The first chapter develops very fully the fundamental probabilities in the theory of insurance against disability, the construction of disability tables, of rates of mortality among the disabled, and so on. A brief account is given of published statistics and of disability tables in use. It is curious that no mention is made of Hunter's *Disability Table*, the one most used in this country.

The second chapter contains an explanation of formulas for disability premiums and reserves under various conditions.

In chapter III there is given an account of the theory and derivation of rates of sickness, of the statistics pertaining thereto, the principal sickness tables, and the methods of calculating net and gross sickness premiums and the mathematical reserves.

Chapters IV and V have to do with applications of the combination of insurance against sickness, disability and death, and also a variety of other forms of insurance relating to marriage, birth, civil responsibility, and so on.

Book II, which deals with collective insurance, gives the technical basis of social insurance and special treatment of insurance against travel accident. An appendix contains an account of the operation of the Caisse Nationale for insurance against death, accident, and to provide for old age. It contains also a brief history and treatment of tontines and societies operating on a tontine basis, and an account with formulas of certain financial operations on a collective basis.

The second volume closes with seven pages of bibliographic references of much value to students of the subjects mentioned therein.

In the opinion of the reviewer these two books cover an unusual variety of insurances, especially the second volume, and the clear and complete development of theory must meet with general approval among mathematicians interested in these subjects. The practical applications are taken chiefly from foreign sources and have little bearing on the methods used in this country.

J. W. Glover

Leçons sur les Invariants Intégraux. By E. Cartan. Paris, Hermann, 1922. x + 210 pp.

The theory of integral invariants was first set forth by H. Poincaré in the third volume of *Les Méthodes Nouvelles de la Mécanique Céleste.* Cartan develops this theory systematically in his Leçons, using a different point of view. Instead of connecting an integral invariant with a system of differential equations as did Poincaré, Cartan considers the integrand of this integral as a differential form and studies its properties of invariance under a group of transformations.

The mathematician will be especially interested to read this book because of the elegant treatment of the subject. Enough of the classical theories of differential forms and continuous groups, for instance, is given to make the book readable. Considerable space is devoted to the development of methods of deriving integral invariants, such as Jacobi's last multiplier and infinitesimal transformations.

These Leçons are to be recommended to students of applied mathematics because of the admirable treatment of physical problems from the modern point of view. The theory of turbulence, the n-body problem, and certain problems in optics are considered from the point of view of tensors and their relation to the integral invariant theory.

E. P. Lane
H. T. Davis

NOTES

The Summer Meeting of the Society will be held at Vassar College, September 6–8, 1923, in conjunction with the meeting of the Mathematical Association of America.

The July, 1922, number of the AMERICAN JOURNAL OF MATHEMATICS (vol. 44, No. 3) contains the following papers: *Plane involutions of order four*, by T. R. Hollcroft; *Differential equations with a continuous infinitude of variables*, by I. A. Barnett; *The factorization of the rational primes in a cubic domain*, by G. E. Wahlin; *On the structure of finite continuous groups with a single exceptional infinitesimal transformation*, by S. D. Zeldin; *The Laplace-Poisson mixed equation*, by K. P. Williams; *The four color problem*, by Philip Franklin.

In the faculty of sciences of the University of Paris, Dr. E. Cartan, professor of rational mechanics, has been placed in charge of the course in analytic and celestial mechanics, Professor P. Montel of the course in rational mechanics, and Professor A. Denjoy, of the University of Strasbourg, of the course in general mathematics. Dr. Cahen has charge of the conferences in rational mechanics.

The Belgian government has granted a subvention to Professor M. Stuyvaert, of the University of Ghent, for the publication of his *Méthodologie mathématique.*

The Berlin Academy of Sciences has awarded its Steiner prize for a memoir on *The determination of all non-degenerate surfaces of the fifth order on which lie one or more series of conics* to Dr. E. G. Togliatti, of the University of Turin. This subject was announced in 1910 (see this BULLETIN, vol. 17, p. 48), but the award was delayed by war conditions.

The Royal Astronomical Society has awarded its gold medal to Professor A. A. Michelson, of the University of Chicago, for his application of the interferometer to astronomical measurements.

This year the University of Texas inaugurates an annual Research Lectureship to be filled by a member of the faculty selected by the Graduate Council for important achievement in research or for distinguished work in the arts. For 1922–1923 the holder of the Lectureship is Professor M. B. Porter. The subjects and dates of the lectures are *The university and research*, March 28; *Measurable functions*, March 29; *Concepts and problems of mathematics*, March 29; *Integration*, March 30.

A campaign for wider distribution is now being promoted by the journal SCIENTIA, which stands for the philosophic synthesizing of science and the intellectual fraternization of peoples. American subscriptions may be sent to Williams and Wilkins, Baltimore, Md., at ten dollars per year. Among contributors to recent and forthcoming issues are the following Americans: W. D. MacMillan, S. Nearing, L. E. Dickson, W. S. Adams, and R. D. Carmichael.

Professor F. Hartogs, of the University of Munich, has been appointed professor of mathematics at the University of Frankfurt a. M.

Sir W. H. Bragg, of the University of London, has been elected a member of the Paris Academy of Sciences in the section of physics.

Professor A. Sommerfeld, of the University of Munich, delivered a course of eight lectures on atomic structure and theories of radiation at the University of California in February, 1923.

Professor E. G. Bill, of Dartmouth College, has been appointed acting dean during the absence of Dean Laycock.

Professor R. P. Stephens of the University of Georgia has been elected president of the Georgia Academy of Science.

At the Georgia School of Technology, Professor Floyd Field has been elected dean of men, and Mr. H. K. Fulmer (not Fuhner, as in this BULLETIN, July, 1922, p. 324) and Mr. E. R. C. Miles have been appointed instructors in mathematics.

At Mississippi College, Clinton, Mississippi, Mr. J. R. Hitt has been promoted to a full professorship of mathematics.

At St. Ignatius College, Cleveland, Ohio, Prof. E. J. O'Leary has been transferred to St. Louis University, and Mr. H. P. Hecken has been appointed professor of mathematics.

Dr. M. A. Nordgaard, of Columbia University, has been appointed professor of mathematics and head of the department at Antioch College, Yellow Springs, Ohio.

Mr. Daniel Hull, of Notre Dame University, has been promoted to an assistant professorship of mathematics.

Miss Lena R. Cole, of the University of Missouri, has been appointed head of the department of mathematics at Hanover College, Hanover, Indiana.

Mr. Burgoyne Griffing, of the University of Kansas, has been added to the staff of the mathematics department of Des Moines University.

At Beloit College, Beloit, Wisconsin, Professor W. A. Hamilton has been appointed chairman of the Administrative Committee, which is in charge of the college during the interim while a new president is being sought to replace President M. A. Brannon, recently appointed Chancellor of the University of Montana. As this work requires all of Professor Hamilton's time, Professor H. H. Conwell has been appointed acting head of the department of mathematics.

Professor V. H. Wells, of Carleton College, Northfield, Minn., has been granted leave of absence and is spending the year teaching mathematics at Williams College.

NEW PUBLICATIONS

I. PURE MATHEMATICS

Andoyer (H.). Tables logarithmiques à treize décimales. Paris, Hermann, 1922.

Aner (H.). Algebraisk formteori i förenklad gestalt. Stockholm, A.-B. Svenska Teknologforeningens Forlag, 1921. 8vo. 105 pp.

Birkemeier (W.). Ueber den Bildungswert der Mathematik. Ein Beitrag zur philosophischen Pädogogik. (Wissenschaft und Hypothese, 25.) Leipzig, Teubner, 1923. 191 pp.

Brandenberger (K.). Didaktik des mathematisch-naturwissenschaftlichen Unterrichts. Zürich, Schulthess, 1920. 8vo. 128 pp.

Burali-Forti (C.) e Marcolongo (R.). Elementi di calcolo vettoriale con numerose applicazioni alla geometria, alla meccanica e alla fisica matematica. 2a edizione, riordenata e ampliata. Bologna, Zanichelli, 1921. 8vo. 19 + 250 pp.

Carnap (R.). Der Raum. Ein Beitrag zur Wissenschaftslehre. Berlin, Reuther und Reichard, 1922. 87 pp.

Engel (F.). See Lie (S.).

Fischer (P. B.). Determinanten. 2te, verbesserte Auflage. (Sammlung Göschen.) Berlin, Vereinigung wissenschaftlicher Verleger, 1921. 8vo. 136 pp.

Heiberg (J. L.). Mathematics and physical science in classical antiquity. Translated by D. C. MacGregor. London, Oxford University Press, 1922. 110 pp.

Hjelmslev (J.). Elementær Geometri. Tredje Bog. København, Gjellerup, 1921. 8vo. 256 pp.

Jung (H. W. E.). Einführung in die Theorie der algebraischen Funktionen einer Veränderlichen. Berlin, Vereinigung wissenschaftlicher Verleger, 1923. 8vo. 6 + 246 pp.

Kommerell (K.). Der Begriff des Grenzwertes in der Elementarmathematik. Leipzig, Teubner, 1922. 12 + 62 pp.

Lie (S.). Gesammelte mathematische Abhandlungen. Band 3: Abhandlungen zur Theorie der Differentialgleichungen. 1te Abteilung. Herausgegeben von F. Engel. Leipzig, Teubner, 1922.

MacGregor (D. C.). See Heiberg (J. L.).

Marcolongo (R.). See Burali-Forti (C.).

Moreux (T.). Pour comprendre la géométrie plane. Paris, Doin, 1922. 252 pp.

Picard (E.). Traité d'analyse. 3e édition, revue et corrigée. Tome 1. Paris, Gauthier-Villars, 1922. 17 + 594 pp.

Study (E.). Die realistische Weltansicht und die Lehre vom Raume. Geometrie, Anschauung und Erfahrung. 2te umgearbeitete Auflage. Teil 1. Braunschweig, Vieweg, 1923. 12 + 85 pp.

Watson (G. N.). A treatise on the theory of Bessel functions. Cambridge, University Press, 1922. 804 pp.

II. APPLIED MATHEMATICS

Alliata (G.). Das Weltbild der Aethermechanik. (Skizze zum Weltbild.) Leipzig, Hillmann, 1922. 8vo. 57 pp.

Becker (W.). Die Relativitätstheorie gemeinverständlich dargestellt. Leipzig, Hachmeister und Thal, 1921. 8vo. 96 pp.

Beer (F.). Die Einsteinsche Relativitätstheorie und ihr historisches Fundament. 7te Auflage. Wien und Leipzig, Perles, 1922. 8vo. 80 pp.

Brose (H. L.). See Reiche (F.).

Charlier (C. V. L.). Introduction to stellar statistics. Lund, Scientia, 1921. 49 pp.

Cohn (E.). Physikalisches über Raum und Zeit. 4te Auflage. Leipzig, Teubner, 1920. 30 pp.

Dallongeville (E.). Le scintillement de l'énergie. Noyon, E. Dallongeville, 1922.

Dopp (P. H.). See Wulf (P. P. T.).

Hatfield (H. S.). See Reiche (F.).

Hoppe (F.). Elektrische Messinstrumente und Messmethoden. Berlin, R. C. Schmidt, 1922. 8vo. 75 pp.

Hort (W.). Technische Schwingungslehre. Ein Handbuch für Ingenieure, Physiker und Mathematiker bei der Untersuchung der in der Technik angewendeten periodischen Vorgänge. 2te, völlig umgearbeitete Auflage. Berlin, Springer, 1922.

Kohnstamm (P.). Lehrbuch der Thermodynamik in ihrer Anwendung auf das Gleichgewicht von Systemen mit gasförmigflüssigen Phasen. Nach Vorlesungen von J. D. van der Waals. 2te, verbesserte Auflage. Teil 1. Leipzig, Barth, 1923. 12 + 287 pp.

Ludewig (P.). Radioaktivität. (Sammlung Göschen.) Berlin, Vereinigung wissenschaftlicher Verleger, 1921. 133 pp.

Mach (E.). Die Prinzipien der Wärmelehre, historisch-kritisch entwickelt. 4te Auflage. Leipzig, Barth, 1923.

von Mecenseffy (E.). Kurvengeometrie des Baumeisters. 6 Hefte. Berlin, W. Ernst, 1920.

Poor (C. L.). Gravitation versus relativity. London, Putnam, 1922. 8vo. 34 + 277 pp.

Reiche (F.). The quantum theory. Translated by H. S. Hatfield and H. L. Brose. New York, Dutton, 1923. 183 pp. $2.50

Stickers (J.). Die wahre Relativitätstheorie der Physik und die Missbegriffe Einsteins. Bielefeld, W. Breitenbach, 1922. 4 + 57 pp.

Thirring (H.). Die Idee der Relativitätstheorie. 2te, durchgesehene und verbesserte Auflage. Berlin, Springer, 1922. 4 + 172 pp.

van der Waals (J. D.). See Kohnstamm (R.).

Wulf (P. P. T.). La théorie de la relativité d'Einstein. Traduit par P. H. Dopp. Bruxelles, Dewett, 1922. 8vo. 96 pp.

Zacharias (J.). Verborgene Gewalten im Weltgeschehen. Eine neue Raum-Kraft-Lehre. München, O. W. Barth, 1922. 8vo. 91 pp.

THE FEBRUARY MEETING OF THE SOCIETY

The two hundred twenty-seventh regular meeting of the Society was held at Columbia University on Saturday, February 24, with the usual morning and afternoon sessions. The attendance included the following thirty-five members: Alexander, Archibald, Barnum, W. J. Berry, Cole, Cowley, Dantzig, Fite, Fry, Gafafer, Glenn, Gronwall, Hausle, Joffe, Kasner, Langman, MacColl, Meder, F. Morley, Mullins, Northcott, Oglesby, Paaswell, Pfeiffer, Reddick, R. G. D. Richardson, Ritt, Seely, Siceloff, Sosnow, Tracey, H. E. Webb, Weisner, Whited, Whittemore.

The Secretary announced the election of the following persons to membership in the Society:

Miss Frances Almira Atwater, Milwaukee-Downer College;
Mr. Heinrich Wilhelm Brinkman, Harvard University;
Professor Robert Ernest Bruce, Boston University;
Professor Joseph Edward Burnam, Simmons College;
Professor Frederick John Dick, Theosophical University;
Mr. Lawrence Murray Graves, University of Chicago;
Mr. Albrecht Wladimir Hanush, Chicago, Ill.;
Mr. Dio Lewis Holl, University of Chicago;
Professor Anna Mayme Howe, Newcomb College;
Professor Emma Louise Konantz, Peking University;
Mr. Bernard Osgood Koopman, Harvard University;
Miss Anna Marguerite Marie Lehr, Bryn Mawr College;
Dr. Earl Emanuel Libman, University of Illinois;
Mr. Robert Bruce Lindsay, Fellow, American-Scandinavian Foundation;
Mr. Renke Gustav Lubben, University of Texas;
Professor Dayton Clarence Miller, Case School of Applied Science;
Mr. Murtach Matthew Sylvester Moriarty, Holyoke, Mass.;
Mr. David Rines, Boston, Mass.;
Limmye Vernon Robinson, Meridian College;
Professor George Wellington Spenceley, Miami University;
Dr. Flora Dobler Sutton, Johns Hopkins University;
Dr. William Phillip Udinski, Texas Agricultural and Mechanical College;
Professor David Locke Webster, Stanford University;
Mr. William Marvin Whyburn, University of Texas;
Miss Constance Wiener, Smith College;
Mr. Walter Wood, Philadelphia, Pa.

Since the Annual Meeting, the following additional members of the London Mathematical Society have entered the American Mathematical Society under the reciprocity agreement:

Mr. John Brill, London;
Professor Horatio Scott Carslaw, University of Sydney;

Professor Robert William Genese, Hastings;
Mr. William Percy Dartrey MacMahon, English High School, Constantinople;
Professor John Raymond Wilton, University of Adelaide.

Twenty applications for membership in the Society were received.

The following committees were appointed:—On Arrangements for the Summer Meeting, H. S. White (chairman), G. M. Conwell, E. B. Cowley, L. D. Cummings, and the Secretary; On Policy and Budget, the President (chairman), Treasurer and Secretary, and E. T. Bell, Henry Blumberg, L. P. Eisenhart, E. R. Hedrick, and A. J. Pell; On Endowment, J. L. Coolidge (chairman), G. E. Roosevelt (secretary), Arnold Dresden, G. C. Evans, and Robert Henderson.

Professor E. V. Huntington was appointed representative of the Society in the Division of Physical Sciences of the National Research Council for the period of three years beginning July 1, 1923, as successor to Professor Veblen.

At the morning session of the Society, Ex-President Frank Morley presided, and at the afternoon session, Professor J. K. Whittemore.

Titles and abstracts of the papers read at this meeting follow below. The papers of Mr. Garabedian, Professor Hollcroft, Mr. Murray, Professor Emch, Dr. Walsh, Dr. Nelson, Professor Schwatt, Dr. Rutledge, Professor Lipka, Mr. Wilder, Mr. Rice, and Professor Kakeya were read by title. Miss Whelan's paper was read by Professor Morley. Miss Whelan was introduced by Professor Morley, Dr. Rainich by Dr. Pfeiffer, and Professor Kakeya by Professor Birkhoff.

1. Mr. C. A. Garabedian: *Circular plates of constant or variable thickness.*

This paper develops a method of series in elasticity, with particular reference to its applications to circular plates of constant or variable thickness. MM. E. and M. Cosserat, in a note in the COMPTES RENDUS, vol. 146 (1908), show how a method of series, that takes account of the geometric parameters that must enter, will yield results in agreement with classical solutions; their applications are confined to cases of constant thickness. In his note on *Circular plates of variable thickness*, PHILOSOPHICAL MAGAZINE, May, 1922, Professor Birkhoff adopts cylindrical coordinates, with the axis of z along the axis of the plate; he then writes $z = \zeta t$, assumes the displacements developable in power series in t, and, by a method of minimizing the energy integral, obtains leading terms when the thickness is variable. The present author uses power series in t, but takes as his starting point the fundamental equations of elasticity. This results in a direct method of series, applicable to cases of variable as well as constant thickness. The method is capable of extension, and affords a solution of a problem which has interested elasticians since Poisson and Cauchy.

2. Professor J. F. Ritt: *Permutable rational functions.*

This paper investigates the circumstances under which two rational functions, $\varphi(x)$ and $\psi(x)$, are such that $\varphi[\psi(x)] = \psi[\varphi(x)]$. A detailed statement of results has appeared in the COMPTES RENDUS for January 8, 1923.

3. Professor T. R. Hollcroft: *The maximum number of cusps of a space curve.*

From a formula for the maximum genus of a space curve of given order n due to Noether, the minimum number of apparent double points is found. Consider the curve projected on a plane from a point of the nodal curve of its developable, or, for n large, from a triple point of this curve. From any point of the nodal curve, two apparent double points project into cusps and from a triple point of this curve, three. The remaining apparent double points project into nodes of the plane curve. The maximum number of cusps that may be added to the singularities of this plane curve is the number that may result from the projection of cusps of the space curve. This gives formulas for the maximum number of cusps of a space curve of given order n lying on a non-singular surface of given order μ. For $\mu = 2$ the maximum number of cusps for a given n is given by formulas involving only n, and this theorem follows: *Every proper space curve with the maximum number of cusps for a given order lies on a quadric surface.*

4. Miss A. M. Whelan: *Theory of the octavic.*

Eight numbers may be arranged into products of differences of the type $\overline{12}\cdot\overline{34}\cdot\overline{56}\cdot\overline{78}$ in 105 ways. These fall into sets of

7 such that all 28 pairs occur. Limiting these sets to those of a certain symmetrical type we find that there are 30 such sets. A sign is attached to each product and the sum of each set of 7 products formed. These sums are the Coble heptads. These 30 heptads fall into 2 sets of 15 such that an even permutation among the 8 numbers leaves each set of 15 unaltered while an odd permutation sends one set into the other. In other words, each set of 15 heptads admits the group $\frac{1}{2}\ 8!$ These sets are referred to as the σ and s heptads.

The following relations are readily proved: $\sum^{15} \sigma_i^n - \sum^{15} s_i^n = 0$ for $n = 1, 2, 3, 4, 5, 6, 8$; $\sum^{15} \sigma_i^7 - \sum^{15} s_i^7 = \mu \sqrt{\Delta}$, where Δ is the discriminant of the octavic whose roots are the 8 quantities, and μ is a constant which is determined by consideration of a particular case. The sums of the powers of the 30 heptads are invariants of the octavic. Invariantive conditions for a triple root are obtained.

5. Dr. G. Y. Rainich: *Analogy between electromagnetic field and analytic function.*

The counterparts of the Cauchy-Riemann equations of Volterra's generalized theory of analytic functions coincide for $n = 4$ with the electromagnetic equations for vacuum in Minkowsky form. The simplest singularity is a singular line which may be identified with an electron path. The analog of the residuum is the electric charge, and its invariance follows from the analog of Cauchy's integral theorem. There is likewise an analogy with the theory of surfaces: the energy tensor is derived from the electromagnetic tensor in the same way as the curvature from the second differential form. If we consider the curvature of the space as produced by the energy tensor, the additional members which arise in one of the sets of the Minkowsky equations by substituting absolute for ordinary differentiation make these equations non-linear in the components of the electromagnetic tensor (appreciable only near the singularities). This non-linearity may account for the equality of charges of all electrons, for the difference between positive and negative electricity, and for the dependence of the electron paths on the field.

6. Professor J. W. Alexander: *The existence of closed geodesics on surfaces.* Preliminary communication.

Associated with a surface S^2, there is a three-dimensional manifold M^3 such that a point of M^3 corresponds to a point of S^2 together with a fixed direction through the point. The geodesics of S^2 correspond to a congruence of curves of M^3, and every closed geodesic of S^2 corresponds to a closed curve of the congruence in M^3. The author proves that the sum of the indices of the closed curves of the congruence is a topological invariant of M^3. From this may be inferred a theorem by Birkhoff on the existence of a closed geodesic on any closed surface of genus 0. The method is extended to the study of the closed geodesics of characteristic surfaces bounded by curves of zero velocity, such as arise in dynamical problems.

7. Mr. F. H. Murray: *The asymptotic expansion of the functions* $W_{k,m}(z)$ *of Whittaker.*

The asymptotic expansion of the functions $W_{k,m}(z)$ defined by Whittaker is already known, if z lies in a sector which does not include the negative half of the real axis; it is shown here that this expansion continues to hold near the negative half of the real axis if the parameters k, m satisfy a certain inequality. An application is made to the study of the "croissance" of the solutions of certain linear differential equations of the second order.

8. Professor Arnold Emch: *Some geometric applications of symmetric substitution groups.*

The geometry of the symmetric group in its fundamental aspects has been investigated by G. Veronese.* As the ultimate object of Veronese was a penetrating study of Pascal's hexagon inscribed in a conic, its generalizations, and of related problems, little attention was given to the various possible invariant curves and surfaces of such groups Since then the literature on problems of invariance of forms under certain groups has grown extensively.

In this paper, the author investigates a number of curves and surfaces which are invariant under the symmetric G_6 and G_{24}, associated with ternary and quaternary spaces respectively. The results obtained may perhaps be indicated by two of the theorems: *All sextics of the* G_6 *which pass through 5 independent sextuples of the* G_6 *pass through a sixth common sextuple. All space sextics of the* G_{24} *form a net; every sextic of the group lies on six cubic cones.*

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* *Interprétation géométrique de la théorie des substitutions de n lettres particulièrement pour n* = 3, 4, 5, 6, *en relation avec les groupes de l'hexagramme mystique*, ANNALI DI MATEMATICA, (2), vol. 11 (1882), pp. 93–236.

9. Dr. J. L. Walsh: *A property of Haar's system of orthogonal functions.*

This note considers the problem of determining an infinite normal orthogonal system of functions $\varphi_1, \varphi_2, \varphi_3, \cdots$ so that, when $\varphi_1, \varphi_2, \cdots, \varphi_n$ are known, the function φ_{n+1} shall be chosen as that normalized function which is orthogonal to $\varphi_1, \varphi_2, \cdots, \varphi_n$ and whose total fluctuation is least. If the functions φ are required to vanish at the ends of the interval considered, the solution of the problem is precisely Haar's system.

10. Dr. C. A. Nelson: *The Riemann adjoints of certain completely integrable systems.*

Completely integrable systems of partial differential equations have been used by Wilczynski and others to investigate the projective differential properties of a surface. For a non-developable surface this system consists of two linear homogeneous equations of the second order in two independent variables. The study of developables requires a system of three equations, two of the preceding type and a third equation of the third order. The author determines all such systems whose adjoints, in the sense of Riemann, also form a completely integrable system. The surfaces whose characterizing systems of differential equations possess this property are of four types. The non-developables are (*a*) non-ruled surfaces whose invariants of lowest order vanish; (*b*) ruled surfaces which have at least one straight line directrix; (*c*) all quadrics. The developables are (*d*) quadric cones.

11. Professor Edward Kasner: *Intermediate curvatures in Riemann space.*

The Riemann curvature of an n-space at a point P depends upon the orientation of a plane or 2-flat through P. Schur proved that if the result is independent of orientation it will also be independent of P. There is then a spherical space. The Ricci curvature depends on a line, or the normal $(n-1)$-flat. If this is to be independent of orientation, the present author in an earlier paper has shown that it must also be independent of P, so the space must obey Einstein cosmological equations. He now defines a curvature depending on a k-flat, which generalizes the earlier concepts. Probably an analog of Schur's theorem is also valid here.

12. Dr. G. A. Pfeiffer: *Concerning the common boundary of two or more regions.*

The author shows that the common boundary of two or more connected regions (sets of inner points) is a continuum, i.e., a closed connected point set. Further, he shows that if the number of regions is greater than two, then the common boundary is a non-decomposable continuum (i.e., a continuum which is not the sum of two proper sub-continua) or the sum of two non-decomposable continua.

13. Dr. Louis Weisner: *Some theorems on insolvable groups.*

The object of this paper is to convert certain well known theorems concerning substitution groups into theorems on abstract groups. Among the theorems thus obtained are the following: If a group G contains a subgroup H of index n in G, such that no subgroup of G which includes H contains a subgroup invariant under G, then the order of every invariant subgroup of G is a multiple of n; if n is not a power of a prime, G is insolvable. If a group G, which is the direct product of G_1 and G_2, contains a maximal subgroup H of index n, which includes no invariant subgroup of G, and if G_1 and G_2 contain two subgroups of H which are of index n in G_1 and G_2 respectively, then G_1 and G_2 are simply isomorphic simple groups.

14. Professor I. J. Schwatt: *The value of* $\sum_{k=1}^{n}\sum_{\alpha=1}^{2k}\tan^p \alpha\pi/(2k+1)$, $\sum_{k=1}^{\infty}\prod_{\alpha=1}^{2k}\operatorname{ctn}^p \alpha\pi/(2k+1)$, *and similar forms in terms of Bernoulli and Eulerian numbers.*

The author obtains formulas connecting the preceding expressions with the Bernoulli and Eulerian numbers.

15. Professor O. E. Glenn: *On the reduction of differential parameters in terms of finite sets.* Preliminary report.

Within a functional domain of rationality defined by derivatives of the function in the arbitrary transformations on the variables, and of a certain quadratic irrationality Δ formed from the latter, the infinitude of covariantive differential parameters, involving nth order derivatives, of a binary quantic, $F = (df)^m = (d\varphi)^m$, is expressible rationally and integrally in terms of arbitrary absolute invariants together with a set of $\frac{1}{2}(n+1)(n+2)(m+1)+2$ differential parameters. This finite set is given explicitly by generalized linear invariant elements and iterated jacobian transvectants formed from such elements.

The problem of complete systems in simpler domains is under consideration.

16. Dr. George Rutledge: *A new form of Stirling's interpolation formula.*

Stirling's interpolation formula is written in a new form, and the relation to the usual form is shown. This formula is a direct consequence of a general formula for the polynomial determined by a set of evenly spaced points (JOURNAL OF MATHEMATICS AND PHYSICS, MASSACHUSETTS INSTITUTE OF TECHNOLOGY, vol. 2 (1922), p. 47) and has many advantages in theoretical investigations.

17. Professor Joseph Lipka: *On the angle between two curves in* V_n.

We consider any two curves c_1 and c_2 in a general curved space of n dimensions issuing from a point P in the same direction, i.e., they have an element of arc $PQ = ds$ in common. If $d\omega$ is the angle between the curves c_1 and c_2 at the point Q, we find that $(d\omega/ds)^2 = k_1^2 + k_2^2 - 2k_1k_2 \cos \theta$, where k_1 and k_2 are the geodesic curvatures of c_1 and c_2 at P, and θ is the angle between the principal geodesic normals to c_1 and c_2 at P. If one of the curves is the geodesic through P in the direction PQ, we have immediately $d\omega/ds = k$, an intrinsic definition of the geodesic curve at a point. Compare the author's paper *Sulla curvatura geodetica delle linee appartenenti ad una varietà qualunque*, RENDICONTI, ACCADEMIA DEI LINCEI, vol. 31, May, 1922.

18. Mr. R. L. Wilder: *On accessibility of the boundary of a domain.*

In a paper presented at the April, 1922, meeting of the Society, the author showed that *in the space constituted by a plane continuous curve,* all points of the boundary of any domain are accessible if the boundary is connected im kleinen. Schoenflies has shown* that *in ordinary euclidean space of two dimensions* all points of the boundary of a plane connected domain are accessible if the boundary is a continuous curve. In the present paper it is shown that both of these results are particular cases of the following theorem: D being either a domain with respect to a continuous curve or a plane connected domain in the ordinary sense, and B its boundary, then in order that any point x of B be accessible from D it is sufficient either that (1) there exist a circle K with center at x, such that the set of all points of B that lie within K is a subset of a connected im kleinen subset $S-D$ (where S is the set of all points in the space considered) or that (2) x belong to no continuum of condensation of B.

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* *Die Entwickelung der Lehre von den Punktmannigfaltigkeiten,* JAHRESBERICHT DER VEREINIGUNG, Ergänzungsband II (1908), p. 215.

19. Mr. L. H. Rice: *Determinantal relations based on a matrix whose elements fall into two classes.*

The author generalizes certain relations published by Muir in 1922,* one of which was based on a matrix in which the elements in a selected rectangular array were assigned to one class and the remaining elements to a second class, while another relation was based on a matrix in which the main diagonal elements were assigned to one class and the remaining elements of the matrix to a second class. Muir's proofs employed respectively Laplace's expansion and Cayley's. The present writer, by making use of a simpler mode of proof, has done away with all restrictions as to the possible distribution of the elements into two classes. The classes may be intermingled in any way, the assignment of an element to the first or second class being independent of that of any other element. Muir has also given a rule for the introduction of certain arbitrary quantities into the matrix; a second rule of the same general nature is given in the present paper.

The results are extended to p-way determinants.

20. Professor S. Kakeya: *Maximum modulus of some expressions of limited analytic functions.*

Continuing investigations previously published† the author shows that of all functions $\{f(x)\}$ which are, in the closed domain $|x| \leqq 1$, analytic and of absolute value not exceeding unity there is one and, except for a constant factor of unit absolute value, only one function $f_0(x)$ such that for $f = f_0$ the expression $\left|\sum_{r=1}^{r=m} \sum_{s=0}^{s=n_r} c_{rs} f^{(s)}(\alpha_n)\right|$, where the c_{rs} are given constants and the α_r are given constants of absolute value less than unity, assumes its least upper bound M. This function $f_0(x)$ is found to be a rational function which on the periphery of its unit circle is of constant absolute value unity.

R. G. D. Richardson,
Secretary of the Society.

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*Edinburgh Royal Society Proceedings, vol. 42, p. 342.

† Tôhoku Imperial University Science Reports, vol. 6 (1917), p. 53.

RELATIONS BETWEEN KINDRED RIEMANNIAN P AND Q FUNCTIONS*

BY D. R. CURTISS

1. *Definitions and Properties of Riemannian Functions.* A Riemannian P function, as the term is used by Klein † and others, is a generalized hypergeometric function, i.e., a solution of a homogeneous linear differential equation of second order having but three singular points, all of which are regular. A Q function is a hypergeometric function which has also apparently singular points, that is, points for which the function is not singular, but which are singular points of the linear differential equation of second order satisfied by the Q function. A linear transformation carries the three regular singular points into the points 0, ∞, 1, and the differential equation of a Q function has then the form, as given in my thesis,‡

$$\frac{d^2Q}{dx^2}+p\frac{dQ}{dx}+qQ=0, \tag{1}$$

where

$$p=\left[\frac{1-\lambda'-\lambda''}{x}+\frac{1-\nu'-\nu''}{x-1}+\sum_{i=1}^{i=k}\frac{1-\sigma_i'-\sigma_i''}{x-s_i}\right],$$

$$q=\frac{1}{x(x-1)}\left[-\frac{\lambda'\lambda''}{x}+\mu'\mu''+\frac{\nu'\nu''}{x-1}+\sum_{i=1}^{i=k}\frac{A_i}{x-s_i}\right].$$

The sum of the constants λ, μ, ν, must be 1, 0, or a negative integer, and, as a consequence of the condition that the points s_i are not to be singular points of solutions, it follows that the σ's are zero or positive integers and the *accessory parameters* A_i verify a system of k equations each of which is quadratic or of higher degree in the A_i.

The symbol

$$Q\begin{pmatrix}\lambda', \mu', \nu', \\ \lambda'', \mu'', \nu'', \end{pmatrix} x$$

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* Presented to the Society, April 14, 1922.

† See especially Klein's lithographed lectures, *Ueber die Hypergeometrische Function.* Both the terms P function and Q function were used by Riemann, *Werke*, p. 67 and p. 323.

‡ *Binary families in a triply connected region*, MEMOIRS OF THE AMERICAN ACADEMY, vol. 13 (1904), No. 1.

is used to denote any Q function which is a solution of (1). The numbers λ' and λ'' are called the *exponents* of the singular point $x = 0$, for reasons familiar in the theory of linear differential equations; μ' and μ'' are the exponents of $x = \infty$; and ν' and ν'' are the exponents of $x = 1$. If the sum of these exponents is $1 - n$, the function is said to be of order n.

For a P function the terms in $(x - s_i)$ disappear from the differential equation, and the sum of the exponents is 1. A P function is, then, a Q function of order zero, and the family of P functions satisfying a differential equation (1) free of terms in $(x - s_i)$ is completely determined by its exponents.

In Gauss's celebrated memoir on special hypergeometric functions,* it is shown that between every three such functions that are kindred, that is, whose corresponding exponents differ by integers or zero, there exists a relation linear and homogeneous in the functions, with coefficients that are polynomials in x. Riemann showed the fundamental importance of the monodromic group in these questions. Gauss's kindred functions are functions that have the same monodromic group. By the use of this conception Riemann gave a new proof of the theorem of Gauss for P functions, and was able to assign an upper limit for the degree of each coefficient.

In my thesis † I have extended this theorem to the case of three Q functions

$$Q_i\begin{pmatrix}\lambda_i', & \mu_i', & \nu_i', & x \\ \lambda_i'', & \mu_i'', & \nu_i'', & \end{pmatrix}, \qquad (i = 1, 2, 3), \tag{2}$$

which are *kindred* in the sense that they have the same monodromic group. In order that this be the case, corresponding exponents must be equal or differ by integers, but this condition alone is not sufficient. If, however, the Q functions belong to *irreducible* families, i.e., families of which no member is a solution of a homogeneous linear differential equation of the first order with single-valued coefficients, they will have the same group if corresponding exponents are equal or differ by integers.

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* *Disquisitiones generales circa seriem infinitam*, WERKE, vol. 3, p. 123.

† Loc. cit., p. 48.

Let us write

$$\begin{aligned}\Delta_1\lambda &= |(\lambda_2' - \lambda_2'') - (\lambda_3' - \lambda_3'')|,\\ \Delta_2\lambda &= |(\lambda_3' - \lambda_3'') - (\lambda_1' - \lambda_1'')|,\\ \Delta_3\lambda &= |(\lambda_1' - \lambda_1'') - (\lambda_2' - \lambda_2'')|,\end{aligned}$$

and similarly for $\Delta_1\mu$, $\Delta_2\mu$, $\Delta_3\mu$, $\Delta_1\nu$, $\Delta_2\nu$, $\Delta_3\nu$. Then if the three families are irreducible and are of order n_1, n_2, n_3, respectively, three kindred functions (2) satisfy an identity

$$\phi_1 Q_1 + \phi_2 Q_2 + \phi_3 Q_3 \equiv 0, \tag{3}$$

where ϕ_1, ϕ_2, ϕ_3 are polynomials of degrees not greater than

$$\begin{aligned}&\tfrac{1}{2}(\Delta_1\lambda + \Delta_1\mu + \Delta_1\nu + n_2 + n_3 - 2),\\ &\tfrac{1}{2}(\Delta_2\lambda + \Delta_2\mu + \Delta_2\nu + n_3 + n_1 - 2),\\ &\tfrac{1}{2}(\Delta_3\lambda + \Delta_3\mu + \Delta_3\nu + n_1 + n_2 - 2),\end{aligned} \tag{4}$$

respectively. To each function Q_1 of the first family there correspond functions Q_2 and Q_3 of the other families which verify (3). From the way in which (3) is derived it follows that ϕ_1 cannot vanish identically unless there is a single-valued function F_1 such that for every function Q_2 of the second family the corresponding Q_3 of the third verifies $Q_3 = F_1 Q_2$. Similarly as regards the identical vanishing of ϕ_2 or ϕ_3.

In particular, we may take Q_2 and Q_3 as P functions, and it follows that every irreducible Q function can be expressed as a linear function with rational coefficients of two P functions. Or again, we have a similar expression for a Q function in terms of a P function and its derivative. The formulas (4) for the degrees of the coefficients ϕ in (3) enable us to supply for two well known theorems regarding relations between kindred P and Q functions proofs that have hitherto been lacking.

2. *A Linear Expression for a Q Function in Terms of a P Function and its Derivative, with Polynomial Coefficients.* In Riemann-Weber's *Die Partiellen Differentialgleichungen der Mathematischen Physik*,* there is given a brief account of some of the properties of P and Q functions. Thus we find on page 44

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* Vol. 2, pp. 40–54.

$$(5)\quad x^{\delta}(1-x)^{\epsilon}Q\begin{pmatrix}\alpha\,, \beta\,, \gamma\,, & \\ \alpha', \beta', \gamma', & x\end{pmatrix} = \overline{Q}\begin{pmatrix}\alpha+\delta, \beta-\delta-\epsilon, \gamma+\epsilon, & \\ \alpha'+\delta, \beta'-\delta-\epsilon, \gamma'+\epsilon, & x\end{pmatrix},$$

where δ and ϵ are any numbers we please. The function $\overline{Q}$ has the same apparently singular points with the same exponents as Q, but not, in general, the same accessory parameters. Also (p. 51), if

$$(6)\qquad P = P\begin{pmatrix}\alpha\,, \beta\,, \gamma\,, & \\ \alpha', \beta', \gamma', & x\end{pmatrix},$$

we have

$$(7)\qquad \frac{dP}{dx} = Q_1\begin{pmatrix}\alpha-1, \beta+1, \gamma-1, & \\ \alpha'-1, \beta'+1, \gamma'-1, & x\end{pmatrix},$$

but here we need the qualification (not mentioned in Riemann-Weber) that none of the exponents for P is zero. Combining (5) with (7), we obtain

$$(8)\qquad x(1-x)\frac{dP}{dx} = \overline{Q}\begin{pmatrix}\alpha\,, \beta-1, \gamma\,, & \\ \alpha', \beta'-1, \gamma', & x\end{pmatrix}.$$

It is then remarked that if $A(x)$ and $B(x)$ are polynomials of degrees $n-1$ and n respectively, with no zeros at 0 or 1, the sum

$$(9)\qquad A(x)x(1-x)\frac{dP}{dx} + B(x)P$$

is in general a Q function of order $2n$ with the exponent scheme

$$(10)\qquad Q\begin{pmatrix}\alpha\,, \beta-n, \gamma\,, & \\ \alpha', \beta'-n, \gamma', & x\end{pmatrix}.$$

We may, however, choose $A(x)$ and $B(x)$ so that (9) is a Q function of order $2n-1$ whose symbol is

$$(11)\qquad Q\begin{pmatrix}\alpha+1, \beta-n, \gamma\,, & \\ \alpha', \quad \beta'-n, \gamma', & x\end{pmatrix}.$$

The question is then raised as to whether *every* Q function whose symbol is (10) or (11) can be expressed in form (9), and a proof is suggested by a count of constants in (9) and in the general Q function (10) or (11). This is unconvincing, however, because the $2n$ effective constants in (9) correspond to $2n$ possible apparently singular points in (10), for example, and the question as to whether the accessory parameters A can be determined is unanswered.

Let us now attack this problem with the aid of (3) and (4), under the restriction that the families concerned are irreducible. In (3), let

$$Q_1 = Q, \qquad Q_2 = -\overline{Q}, \qquad Q_3 = -P,$$

as in (10), (8) and (6). Then all the numbers $\Delta\lambda$, $\Delta\mu$, $\Delta\nu$ are zero, and the degrees of ϕ_1, ϕ_2 and ϕ_3 are less than or equal to 0, $n-1$ and n, respectively. Thus we have

$$(12) \qquad CQ = \phi_2(x)x(1-x)\frac{dP}{dx} + \phi_3(x)P,$$

where the degrees of ϕ_2 and ϕ_3 are not greater than $n-1$ and n respectively. The constant C cannot vanish, since a relation

$$P = F_1 x(1-x)\frac{dP}{dx},$$

where F_1 is a single-valued function of x, cannot be verified by all the members of a P family. Division of (12) by C gives the desired representation for the Q function (10) by the expression (9). The same result follows if the Q function (11) is used instead of (10).

This representation holds even when one or more of the exponents in (6) is zero. To prove this we consider the families

$$x^\delta(1-x)^\epsilon P,$$
$$x^\delta(1-x)^\epsilon Q,$$

where P has the symbol (6) and Q the symbol (10) or (11), the numbers δ and ϵ being so chosen that in the new families no exponent is zero. For these functions (12) becomes, after division by C,

$$x^\delta(1-x)^\epsilon Q = \phi_2(x)x(1-x)\frac{d}{dx}[x^\delta(1-x)^\epsilon P] + \phi_3(x)x^\delta(1-x)^\epsilon P,$$

where ϕ_2 and ϕ_3 are polynomials in x of degrees not greater than $n-1$ and n respectively. If we carry out the indicated differentiation, rearrange terms and divide by $x^\delta(1-x)^\epsilon$, we have

$$Q = \phi_2(x)x(1-x)\frac{dP}{dx} + \{\phi_2(x)[\delta(1-x) - \epsilon x] + \phi_3(x)\}P,$$

which is of the form desired.

There remains the question whether we can dispense with the assumption that the families are irreducible. In my thesis I have shown that there are reducible Q families which do not have the same group as *any* P family. Obviously no such Q *family* (10) or (11) can be expressed in form (9), but particular members might be so expressed in terms of particular P functions. However, I have shown in my thesis that formula (3) holds for any three families having the same group, and (4) holds also when there are no semisingular points. Hence the representation we are discussing holds even in reducible cases provided Q and P have the same group and have no semisingular points.

3. *A Linear Expression for a Q Function in Terms of P Functions, with Constant Coefficients.* The other theorem, for which I have found no previous proof other than a count of constants, equally unconvincing, is that of Klein * which states that to form a Q function of order κ in the most general way we must make a linear combination, with constant coefficients, of $\kappa + 1$ kindred P functions. The statement is somewhat ambiguous; however, we shall now show that it is true, in the sense that *every* Q function is so expressible, with reservations in certain reducible cases as in § 2. This answers a question raised on page 49 of my thesis.

We will first suppose the Q family irreducible. If of even order we use the notation (10), if of odd order, (11), and express Q, by formulas (3) and (4), in terms of the two kindred P families, P and $\bar{P}$, where P has the symbol (6), and $\bar{P}$ the symbol

$$\bar{P}\begin{pmatrix} \alpha + 1, \beta, \gamma, \\ \alpha' - 1, \beta', \gamma', \end{pmatrix} x . \tag{13}$$

This gives us

$$CQ = \phi_2(x)\bar{P} + \phi_3(x)P, \tag{14}$$

where C is a constant, and ϕ_2 and ϕ_3 are polynomials in x of degrees not greater than $n - 1$ and n respectively in case Q is of form (10), and of degrees not greater than $n - 1$ each if Q is of form (11). The form of the exponents at zero shows that $\bar{P}$ cannot be equal to P multiplied by a single valued function F, hence C is not zero and we can divide through by C. A typical term of the product $\phi_3(x)P$ is

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* *Hypergeometrische Funktionen*, p. 233.

$$(15)\quad a_k x^k P\begin{pmatrix}\alpha\,, \beta\,, \gamma\,, & x\\ \alpha', \beta', \gamma', & \end{pmatrix} = a_k P_k\begin{pmatrix}\alpha + k, \beta - k, \gamma\,, & x\\ \alpha' + k, \beta' - k, \gamma', & \end{pmatrix},$$

where a_k is a constant; similarly the product $\phi_2(x)\bar{P}$ is a sum of terms $b_k\bar{P}_k$. Thus Q is expressed in the desired form

$$(16)\qquad Q = \sum_{k=0}^{n-1} b_k\bar{P}_k + \sum_{k=0}^{n} a_k P_k$$

if Q is of order $2n$, or

$$(17)\qquad Q = \sum_{k=0}^{n-1} b_k\bar{P}_k + \sum_{k=0}^{n-1} a_k P_k$$

if Q is of order $2n - 1$, where the a's and b's are constants and the functions $\bar{P}_k$ and P_k are P functions as indicated in (15).

As in § 2, we need additional hypotheses in case Q is reducible. Our proof is then not valid unless the families Q, P, $\bar{P}$ have the same group (which does not follow merely because corresponding exponents differ by integers) and no point is semisingular. In my thesis (p. 59) I have considered the exceptional cases for a Q function of order 1, and have shown that such a Q function can always be expressed linearly in terms of two kindred P functions if its monodromic group is possible for P functions. This proof requires the examination of various types and classes of reducible groups, and we shall not attempt here to discuss the corresponding problem for Q functions of order higher than 1.

As a consequence of Klein's theorem it follows that every $\kappa + 2$ irreducible Q functions of order κ having the same regular singular points and the same exponents at these points, but whose accessory parameters may differ, are linearly dependent; and this conclusion holds in certain reducible cases.

These theorems and methods of proof admit generalizations to the solutions of homogeneous linear differential equations of order n.

Northwestern University

THE REDUCTION OF SINGULARITIES OF PLANE CURVES BY BIRATIONAL TRANSFORMATION

PRESIDENTIAL ADDRESS DELIVERED BEFORE THE AMERICAN MATHEMATICAL SOCIETY, DECEMBER 28, 1922

BY G. A. BLISS

1. *Introduction.* In the domain of mathematics there are a number of famous theorems which have stimulated the keen interest of mathematicians over extended periods of time, and whose proofs have presented a long continued challenge to the powers of mathematical logic. To some of these, in spite of the fact that the theorems themselves remain unproved, our science is indebted for important advances.

As one illustration I may mention the well known theorem of Jordan which states that a simply closed continuous plane curve divides the plane into two and only two connected regions. This is a theorem of whose truth we are convinced at the start intuitively. But what we may for the moment call intuitive reasoning busies itself with simpler cases only, and is impatient of exceptions and refinements, unless by long study and continuous contemplation it has become a very sophisticated intuition indeed. Jordan was the first to insist that the theorem needed proof. Since his initial effort many others have attempted to give the conclusion of the theorem a substantial logical basis, and the theories of point sets and of functions of a real variable have been greatly enriched thereby. I am particularly interested to mention this theorem because it seems to me that with regard to it a satisfactory conclusion has been reached. There is doubtless still opportunity for improvements and simplifications in its proofs, but some at least of them have stood the test of examination by widely scattered experts. It is encouraging to have this evidence that not all of the mathematical questions generally recognized as most difficult are impossible to answer.

One should mention, of course, among the notable illustrations of the type of theorem which I have been discussing, Fermat's last theorem and the so-called four-color map theorem. The latter asserts that every map can be colored with four different colors in such a way that each country has one color only, and no two countries adjoining along a boundary have the same color. No one has constructed a map which can not be so colored, and most of us, I think, believe that the theorem is true. But so far as I know no proof of it has as yet been widely accepted, the vote usually standing one, namely the author, strongly in favor, many others doubtful, and some, who have had the temerity to examine in detail the screed in question, definitely against. This state of affairs is even more pronounced in the case of Fermat's last theorem, with the difference that one may with justice, it seems to me, feel very uncertain of the validity of the theorem itself. All of the proved evidence is in its favor, but the behavior of very large numbers, far without the range of our every-day experience, is so beyond our ken that it should not be a surprise to us if some day it were found that the theorem is not true. We are fortunate in some ways, however, that these questions remain unanswered. For every attack by experts upon them is likely to yield, as has been true notably in the past, important advances in the domains of analysis situs and the theory of numbers. The chapter of progress associated with Jordan's theorem may still be subjected to improvements and additions, but as compared with those in which the centers of interest are the four-color map theorem and Fermat's last theorem, it is relatively completed and closed.

Many of you will have inferred from the title of this address the theorem of which I wish to speak to you especially to-day. It is the one which states that by a birational transformation an algebraic curve, no matter how complicated its singularities may be, can always be transformed into another having only double points with distinct tangents. It is with some diffidence that I designate this theorem as one of the type of which I have hitherto been speaking, since among my mathematical friends there are some for whose opinions I have great respect who disagree with me as to the difficulty of its proof. Unfortunately I have not been near enough to them to find out with completeness what is in their minds. My purpose this afternoon will be achieved, however, if I can interest you again for the moment in some of the work of the sequence of distinguished mathematicians who have studied the theorem, and if I can make clear to those of you who have not specialized in the subject, some of the difficulties which they have sought to overcome. Up to the present time I do not know of any proof of the theorem which is sufficiently simple and unsophistocated to justify one at least of its important applications in the theory of algebraic functions to which I shall presently refer again.

2. *The Two Transformation Theorems.* In the statement of the theorem mentioned in § 1, reference is made to the notion of a birational transformation. There are two types of birational transformations, one of which is well illustrated by the very simple but important example $\xi = x$, $\eta = x/y$ relating the points of the xy-plane to those of the $\xi\eta$-plane. If one prefers homogeneous coordinates this transformation may be written in the form $\rho\xi_1 = x_2x_3$, $\rho\xi_2 = x_3x_1$, $\rho\xi_3 = x_1x_2$. It establishes a one-to-one correspondence between the points of the two planes with the exception of those on certain loci, in this case the coordinate axes, and is a special example of the general class of birational transformations which establish such correspondences between two planes and which are called Cremona transformations. Incidentally the transformation transforms every not too special algebraic curve $f(x, y) = 0$ of the xy-plane into a similar curve $\varphi(\xi, \eta) = 0$ of the $\xi\eta$-plane and establishes a one-to-one birational correspondence between their points. It is also possible, however, to have a one-to-one correspondence between the points of two curves $f(x, y) = 0$, $\varphi(\xi, \eta) = 0$, with coordinates (x, y) and (ξ, η) of corresponding points rationally expressible in terms of each other, but such that the correspondence is not a part of any Cremona transformation between the planes of the two curves such as has been described above. Such a transformation has been called a birational transformation relating the two curves alone. A simple example of such a transformation is defined by the

equations $\xi = x^2$, $\eta = y$ when applied to a curve not symmetric with respect to the y-axis.

Before proceeding to the discussion of the various proofs which have been made of the theorem I should like to say that there are really two closely related theorems which have been considered in this connection. To understand them we should remember that an "ordinary" singular point of a curve is one through which pass a finite number of branches whose tangents are distinct. For such a point $(x, y) = (a, b)$ in the finite part of the plane these branches will be determined analytically by convergent series of the form

$$x = a + \alpha t + \cdots, \qquad y = b + \beta t + \cdots,$$

in which the coefficients α and β are not both zero, and the different branches through (a, b) will have distinct ratios $\alpha : \beta$. The two theorems mentioned are then as follows.

THEOREM I. *Every irreducible algebraic plane curve can be transformed into another which has only ordinary singular points by a Cremona transformation of the planes of the two curves.*

THEOREM II. *Every irreducible algebraic plane curve can be transformed into another which has only ordinary double points by a transformation which is birational between the two curves.*

Theorem II is the one of which I have already spoken, and I may say here that there are two types of proofs of it. For one of them the conclusion of Theorem I is assumed, and a further birational transformation is then determined which will replace ordinary multiple points of multiplicity higher than two by ordinary double points. The other type of proof is more direct, presenting at once a birational transformation producing ordinary double points only, without the use of Theorem I.

In the theory of algebraic functions these two theorems have an important application. The genus of an algebraic curve, and the more important properties of the Abelian integrals and the field of rational functions associated with the curve, are invariant under birational transformations. The presentation of the theory, from the standpoint of the so-called transcendental methods inaugurated by Riemann and developed and amplified by many other writers, is greatly simplified if one can presuppose that the curve under consideration has only ordinary singular points. Some further advantage in simplicity would be gained if only ordinary double points were present, but, as one of my mathematical colleagues has aptly said, this would be in the nature of a luxury rather than a necessity. It is significant, however, that most lecturers and writers who approach the subject from the standpoint of Riemann are content to start with the result of Theorem I, which they probably would not do if they found that Theorem II were provable in relatively as simple and elementary a fashion. Some of the other theories of algebraic functions which do not presuppose any special properties of the singular points of the basal curve seem to me highly interesting and desirable, notably that of Dedekind and Weber as amplified and presented by Hensel and Landsberg.* Theoretically this is certainly a very elegant and satisfying method of approach.

In speaking a few minutes ago of the correspondence established between two algebraic curves by a birational transformation, I used locution of which I very much disapprove, but which at that moment could not very well be explained. The elements of an algebraic curve which are set into one-to-one correspondence with those of another curve by a birational transformation are not the points of the curve, but the so-called branches or cycles. Through each finite point of the curve there passes one of these branches, if the point is simple, but possibly more than one if the point is multiple, and there are besides branches belonging to the so-called points at infinity. Analytically each branch is determined in non-homogeneous coordinates by equations of the form $x = P(t)$, $y = Q(t)$, where $P(t)$ and $Q(t)$ are convergent power series having usually only terms with positive powers of t, but for a finite number of branches having also a finite number of terms with negative exponents. If homogeneous coordinates are used the cycle or branch can always be represented by series having only positive powers of t. Weierstrass especially emphasized the point of view that a curve may well be thought of as the totality of the branches which satisfy its equation, and I think that failure to appreciate his contention has in some cases caused confusion in the proofs of the theorems of which I have been speaking. Birational transformations do not in general transform curves point for point into each other, but they do establish a unique correspondence between the branches of the two curves.

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* *Theorie der algebraischen Funktionen einer Variabeln*, 1902.

3. *Geometric Interpretations.* The most useful geometric picture of an algebraic equation $f(x, y) = 0$ which we have is its so-called Riemann surface. The effectiveness of the picture is due to the fact that there is a one-to-one continuous correspondence between the points on the surface and the branches which satisfy the equation. Furthermore the character of the cycle of sheets of which each point of the surface is a vertex indicates important properties of the corresponding analytic branch. Besides the Riemann surface, however, there are two other geometric representations of an algebraic curve which have especial importance in connection with the transformation theorems stated above, but each of which is open to the objection that it visualizes for us only the real branches of the algebraic curve in question.

For the first of these, consider the pencils of lines through two points A and B selected arbitrarily in the plane. In the pencil through A, for example, we may regard each ray as designated by the value x of its anharmonic ratio with three fixed rays through A chosen in advance; and the rays through B are similarly determined by numbers y. If we now associate with each pair of values (x, y) satisfying an algebraic equation $f(x, y) = 0$ the point of intersection of the corresponding x- and y-rays through A and B, we obtain a curve representing geometrically the real branches of our equation. In order to give equal consideration to all the rays through A and B without using infinite values for the coordinates we may introduce the homogeneous variables x_1, x_2, and y_1, y_2 defined by the equations $x = x_1/x_2$, $y = y_1/y_2$.

The other geometric representation of the equation $f(x, y) = 0$ is so familiar that I need not speak of it in detail. If we replace x and y by homogeneous coordinates x_1, x_2, x_3 defined in the usual way by the equations $x = x_1/x_3, y = x_2/x_3$, then the equation $f(x, y) = 0$ determines a curve in the projective plane as soon as a coordinate triangle and unit point have been assigned.

For the sake of convenience let us agree to call the former of these two geometric interpretations of the equation $f(x, y) = 0$ the function-theoretic one, and the latter simply the geometric interpretation. The reason for this nomenclature is that the former is closely allied to the theory of algebraic functions, while the latter is that of ordinary plane projective geometry. The function-theoretic interpretation is the more convenient one for representing properties of curves which are invariant under transformations of the form

$$(1) \qquad x = \frac{a_1\xi + a_2}{a_3\xi + a_4}, \qquad y = \frac{b_1\eta + b_2}{b_3\eta + b_4},$$

while the geometric interpretation has similar advantages for transformations

$$(2) \qquad x = \frac{a_1\xi + b_1\eta + c_1}{a_3\xi + b_3\eta + c_3}, \qquad y = \frac{a_2\xi + b_2\eta + c_2}{a_3\xi + b_3\eta + c_3}.$$

We say in either case that a point of the curve $f(x, y) = 0$ with finite or infinite coordinates is an ordinary double point if it has this character after an appropriate transformation (1) or (2) into a non-special point with finite coordinates. The points on the line AB in the function-theoretic representation are special points.

It is perhaps more satisfactory to define analytically what is meant by a curve with ordinary double points only. In the function-theoretic case the equations of a branch can always be taken in the form

$$x_i = a_i + \alpha_i t + \cdots, \qquad y_i = b_i + \beta_i t + \cdots, \qquad (i = 1, 2),$$
$$(a_1, a_2) \neq (0, 0), \qquad (b_1, b_2) \neq (0, 0),$$

and the branch is said to be linear if the two determinants

$$a_1\alpha_2 - a_2\alpha_1, \qquad b_1\beta_2 - b_2\beta_1$$

do not both vanish. A second branch with primed coefficients has the same center if

$$a_i' = ka_i, \qquad b_i' = lb_i, \qquad (i = 1, 2);$$

and two branches with the same center are said to have distinct tangents if the expression

$$(a_1\alpha_2' - a_2\alpha_1')(b_1'\beta_2 - b_2'\beta_1) - (a_1'\alpha_2 - a_2'\alpha_1)(b_1\beta_2' - b_2\beta_1')$$

is not zero.

In the geometric case a branch has equations

$$x_i = a_i + \alpha_i t + \cdots, \quad (i = 1, 2, 3), \; (a_1, a_2, a_3) \neq (0, 0, 0),$$

and is linear if the determinants $a_i\alpha_k - a_k\alpha_i$ $(i \neq k)$ are not all zero. A second branch with primed coefficients has the same center if $a_k' = \rho a_k$ $(k = 1, 2, 3)$; and two branches with the same center have distinct tangents if the determinant $(a_1\alpha_2\alpha_3')$ is different from zero. For branches with finite centers $(x, y) = (a, b)$ these definitions of branches with distinct tangents reduce to those of the usual cartesian analytic geometry. A curve with no singularities except ordinary double points is in either interpretation one such that all of its branches are linear, such that none of its points is a center of more than two of its branches, and such that every pair of its branches having a common center has distinct tangents. These properties are invariant in the function-theoretic and geometric cases under the respective transformations (1) and (2).

Now it happens that the branches which satisfy an equation $f(x, y) = 0$ may determine a curve with ordinary double points only in one of these interpretations, while more complicated singularities are present in the other. This is due to the fact that we have added in two different ways points with infinite coordinates to the totality of finite points (x, y) in the plane in order to obtain the two interpretations. If, for example, the line $x = \infty$ through A is intersected by n distinct branches in the function-theoretic case, then the curve for the geometric interpretation of the equation will have an ordinary singular point with n branches at the intersection of the x-axis with the line at infinity. Similarly if the line at infinity in the projective plane intersects the geometric curve in n points distinct from each other and from the x- and y-axes, then the function-theoretic curve will have an ordinary singular point with n branches through the intersection of the lines $x = \infty$ and $y = \infty$.

We could readily analyze this correspondence in greater detail, but the essential thing to remark is that there is really a pair of Theorems II which are not equivalent. The distinction between these theorems has not, in my opinion, been sufficiently emphasized in the literature, though proofs applying to both have been presented. At the present time the function-theoretic theorem appears to me somewhat easier to handle, though neither of them is simple.

4. *Noether's Theorem.* Those of you who have not been specialists in this domain will perhaps be interested to know something of the origins of these theorems. Theorem I is, with what seems to be universal agreement, ascribed to Noether. In 1871 * he stated the theorem and indicated the proof which has since been modified and elaborated in one way or another by many writers. A difficulty in the proof not explained in this original paper can be removed by methods used in the same year by Hamburger † in his study of the Puiseaux expansions at a singular point. Noether ‡ himself in 1876 treated the question more elaborately in a paper devoted to the analysis of the singular points of an algebraic curve. Bertini § was dissatisfied with the methods used by Noether, and in 1888 gave a direct proof of the theorem which he characterizes as simple and rigorous. The proof with which I am most familiar is that given by Picard || and ascribed by him to Simart. The geometers would be perhaps most interested in those of Severi ¶

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* GÖTTINGER NACHRICHTEN, 1871, page 267.

† ZEITSCHRIFT FÜR MATHEMATIK UND PHYSIK, vol. 16 (1871), p. 461.

‡ MATHEMATISCHE ANNALEN, vol. 9 (1876), p. 166, especially § 5.

§ LOMBARDO RENDICONTI, (2), vol. 21 (1888), page 326. In commenting on Noether's paper he says "le considerazioni (§ 5) lasciano forse a desiderare maggiore semplicita e chiarezza," and he says that his intention in his own paper is to prove Noether's theorem "semplice e rigorosa."

|| *Traité d'Analyse*, vol. 2, 1st ed., 1893, page 360; 2d ed., 1905, p. 404.

¶ *Lezioni di Geometria Algebrica*, 1908, p. 61.

and Enriques,* but there are many others.† The proofs are all modifications of Noether's original one, and there is no question with regard to the validity of the theorem. I may add that Halphen ‡ gave two methods of transforming a curve into one with ordinary singular points only by birational transformations which are not Cremona transformations.

The method of the proof is to place the origin of coordinates at one of the singular points of the curve, say one of order k, and then to apply the simple quadratic transformation $\xi = x$, $\eta = x/y$ of which I have spoken above. In the new curve the character of the singular points not at the origin remains the same as before, but the singularity of order k at the origin is exploded by the transformation, and when one examines the pieces remaining two new ordinary singular points are found at corners of the coordinate triangle, plus one or more other fragmentary singular points on one side of this triangle the sum of whose orders is k. The difficulty which I have mentioned above and which was not originally explained by Noether lies in the possibility that there may be only one such fragment of the same order as the original singularity, in which case the situation might easily be as bad after as before the transformation. It is provable, however, that after a finite number of repetitions of the transformation the number of the fragments will surely be greater than one, and their orders each less than k. By a continuation of the process, therefore, one can finally replace the singular point originally placed at the origin, and indeed all of the singular points, by ordinary singular points only. It is not surprising that there have been so few variations from the method of proof originally proposed by Noether, if one remembers the theorem that every Cremona transformation is decomposable into a succession of projective transformations and simple quadratic transformations of the type we have been considering.

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* *Lezioni sulla Teoria Geometrica delle Equazioni e delle Funzioni Algebriche*, vol. 2, 1918, p. 417.

† See ENCYCLOPÄDIE, III C 4, p. 363; Pascal, *Repertorium der Höheren Mathematik*, vol. 2, 2d ed., p. 291; Brill und Noether, *Die Entwicklung der Theorie der algebraischen Functionen*, JAHRESBERICHT DER VEREINIGUNG, vol. 3 (1892–3), p. 369. Interesting references which do not seem to be listed in these places are Jordan, *Cours d'Analyse*, vol. 1, 2d ed., 1893, p. 588; and Clebsch-Lindemann, *Vorlesungen über Geometrie*, vol. 1, 1876, p. 491.

‡ COMPTES RENDUS, vol. 80 (1875), p. 638, and OEUVRES, vol. 1, p. 358; JOURNAL DE MATHÉMATIQUES, (3), vol. 2 (1876), p. 87.

5. *The Kronecker Group of Proofs of Theorem* II. With regard to Theorem II, it seems to me that the story is quite a different one. The origin of the theorem is less easy to determine, and no proof of it has been adopted with universal or even with widespread approval. During a somewhat cursory examination of the literature over a year ago I found so many doubtful comments by authors themselves upon proofs which had preceded theirs, and so many questions of my own which it seemed difficult to answer, that I resolved to try by a closer personal examination to convince myself of the merits of some of the published demonstrations of the theorem. It is of the impressions gained from this study that I wish to speak to you now. After we have looked for a moment at the various types of proofs which have been given it will perhaps be easier to describe intelligibly the sources from which the theorem seems to have been developed.

The proofs fall into four main groups,* and I shall speak first of that which originated with Kronecker because it is the one in which I have myself been primarily interested. In 1881 Kronecker † published a paper in which he discusses the factorization of the discriminant of an algebraic function, and which he had presented to the Berlin Academy many years before, in 1862. It is a very well known paper, for many reasons one of the most suggestive and interesting which I have ever read. In it he shows that the discriminant of an algebraic function y of x, defined by an algebraic equation $f(x, y) = 0$, has always the form $D = R^2\Delta$, where R and Δ are two polynomials in the variable x. If the equation $f(x, y) = 0$ is subjected to a birational transformation of the form

$$\xi = x, \qquad \eta = r(x, y)$$

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* For the literature see Brill and Noether, loc. cit., pp. 369 ff.; Wirtinger, ENCYCLOPÄDIE, II B 2, p. 127; Berzolari, ibid., III C 4, pp. 362 ff.; and in Pascal's *Repertorium*, loc. cit.

† JOURNAL FÜR MATHEMATIK, vol. 91 (1881), p. 301.

upon the dependent variable alone, then the resulting equation $\varphi(x, \eta) = 0$ will have a discriminant $D_1 = R_1^2\Delta$ in which R_1 is in general different from R, while the factor Δ remains unchanged. A principal purpose of the paper is to determine the birational transformation in such a way that the roots of R_1 are distinct from each other and from those of Δ. If this has been accomplished, then it is provable that the curve $\varphi(x, \eta) = 0$ can have no singularities in the finite part of the $x\eta$-plane except ordinary double points. This last step I do not find explicitly explained in Kronecker's paper, though it is clear from other sources that he and his followers were well aware of the geometric significance of his transformation of the discriminant.

Kronecker states that the results of the paper of which I have been speaking were in part known to him as early as 1857, and that he used them repeatedly in his lectures for many years before the paper itself was published. His transformation of the discriminant was adopted by Weierstrass in lectures on the theory of algebraic functions in 1869. I am acquainted with Weierstrass' treatment of the question only through a brief outline by Brill and Noether * and a more detailed paper based upon this outline published by Thomé in 1903.† The papers of Kronecker and Thomé are both very clear and convincing, but neither of them affords a complete proof of either the function-theoretic or the geometric Theorem II because after their transformations there may still be complicated singularities at infinity, though only ordinary double points in the finite part of the plane.

In order to secure his birational transformation simplifying the discriminant, Kronecker made use of the rational functions $\eta(x, y)$ on the Riemann surface of the curve $f(x, y) = 0$ which have no poles on the finite part of the surface, the so-called integral algebraic functions in the field of rational functions associated with the curve. A general formula for such a function is

$$\eta(x, y) = u_1(x)\eta_1(x, y) + \cdots + u_n(x)\eta_n(x, y), \tag{3}$$

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* Loc. cit., pp. 375–6.

† JOURNAL FÜR MATHEMATIK, vol. 126 (1903), p. 52.

where $\eta_1, \cdots, \eta_n$ is a suitably selected set of linearly independent functions of this sort, and the coefficients $u_1, \cdots, u_n$ are arbitrary polynomials in x. If these coefficients u_k are suitably specialized, then x and η will satisfy an algebraic equation $\varphi(x, \eta) = 0$ birationally related to $f(x, y) = 0$ and with a discriminant of the desired simplicity.

In 1902 Hensel and Landsberg * extended this reasoning of Kronecker. The functions $\eta(x, y)$ which Kronecker uses are restricted to have minimum orders zero at the finite points of the Riemann surface of $f(x, y) = 0$, but are unrestricted at the infinite points. A formula like (3) still holds, however, if the functions $\eta(x, y)$ have the properties just mentioned with the exception that minimum orders different from zero, either positive or negative, are admitted at a limited number of finite points of the Riemann surface. If now the number of poles allowed for $\eta(x, y)$ is sufficiently large, and if the coefficients $u_1, \cdots, u_n$ are again suitably specialized, the function $\eta(x, y)$ will not only satisfy an equation $\varphi(x, \eta) = 0$ birationally related to $f(x, y) = 0$, and have a discriminant of the form attained by Kronecker, but it will also have finite distinct values at the infinite points of the Riemann surface. This brings us to the function-theoretic Theorem II provided that the poles allowed for $\eta(x, y)$ are of order one and over distinct values of x. For if on the Riemann surface of $\varphi(x, \eta) = 0$, which is the same as that for $f(x, y) = 0$, the function $\eta(x, y)$ has distinct finite values at the value $x = \infty$ and simple poles over distinct finite values of x, then the infinite branches of $\varphi(x, \eta) = 0$ give rise to no multiple points in the function-theoretic interpretation of the equation. If furthermore the discriminant of $\eta(x, y)$ has the simple form attained by Kronecker, then the curve $\varphi(x, \eta) = 0$ can have no singularities at finite points (x, η) except ordinary double points.

The discussion of the separation of the roots of the discriminant given by Hensel and Landsberg seems to me at one point incomplete † without the addition of some

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* *Theorie der algebraischen Funktionen einer Variabeln*, pp. 402–9.

† Loc. cit., the paragraph on pages 407–8.

details from Kronecker's original memoir. Their final statement of the theorem does not distinguish between the function-theoretic and geometric cases upon which I have insisted above, and it is followed by the assertion without a reference that the theorem was first proved by Kronecker. So far as I can discover from published papers, this last statement is not quite just to themselves. The extension of Kronecker's proof which enables one to secure proper behavior of the transformation at infinite as well as finite points of the Riemann surface is apparently their own, and it is a much more difficult extension than is indicated by my brief outline of their argument.

I have been interested to attempt to prove the geometric Theorem II also by Kronecker's method, and have found a proof which seems to me satisfactory. If an algebraic equation $\varphi(\xi, \eta) = 0$ of degree q defines q expansions for η at $\xi = \infty$ of the form

$$\eta = \alpha_k\xi + \beta_k + \gamma_k \frac{1}{\xi} + \cdots, \qquad (k=1, \cdots, q; \alpha_k \neq \alpha_l \text{ for } k \neq l),$$

and if the discriminant for η has the simplified form attained by Kronecker, then it is provable that in the projective plane the corresponding curve meets the line at infinity in distinct points and has no singularities in the finite part of the plane except ordinary double points. To obtain such a curve birationally related to $f(x, y) = 0$ one can start by selecting $q > 2p + 2$ ordinary points on the Riemann surface of $f(x, y) = 0$ over distinct values of x, p being the genus of the surface. The functions $\eta(x, y)$ which have no singularities except possibly simple poles at these points constitute a linear family of the form

$$(4)\quad \eta(x, y) = c_1\eta_1(x, y) + \cdots + c_\nu\eta_\nu(x, y), \qquad (\nu = q - p + 1),$$

the coefficients c_k being constants. A suitable special choice of these coefficients can easily be made so that the resulting function $\xi(x, y)$ has surely a simple pole at each of the q points. With more difficulty a second special function $\eta(x, y)$ of the family can then be chosen which with ξ satisfies an equation $\varphi(\xi, \eta) = 0$ birationally related to $f(x, y) = 0$ and which has the further properties described above.

Here, then, based upon the methods of Kronecker, are proofs of both the function-theoretic and geometric Theorems II. They are what I should call sophisticated proofs, since they make use of properties of rational functions with prescribed poles on the Riemann surface of $f(x, y) = 0$, which are well known and well established but not elementary. These properties are proved in the theory of algebraic functions as presented by Hensel and Landsberg without restrictive assumptions upon the singularities of $f(x, y) = 0$. If the so-called transcendental method of Riemann is adopted, however, it is usually presupposed in proving them that the singularities of the curve $f(x, y) = 0$ are ordinary multiple points only, or, in other words, that the result of Theorem I has been attained by transformation. An easily derived property which I have found for the linear families (4) aids very much in simplifying the proofs of both the function-theoretic and geometric interpretations of Theorem II.

6. *The Halphen Group of Proofs of Theorem* II. There is a second very interesting series of discussions of Theorem II which seems to have been inaugurated by Halphen in 1884.* So far as I have been able to discover this paper of Halphen's contains the first explicit statement and proof of Theorem II which has been published. In 1893 Picard † reproduced Halphen's argument. At its conclusion, in commenting on a particular step in the proof, he says that the point is not to be doubted, but that he does not see quite the means of establishing it so as to avoid every objection. The reasoning by continuity which he has just been using makes the accuracy of the conclusion seem to him more than probable. In a footnote he adds that Halphen arrives at the theorem without mentioning the difficulty.

This comment of Picard seems to have instigated a series of attacks on the theorem which continued for more than a decade. In the same year, 1893, Simart * modified Halphen's proof as presented by Picard and remarks that the theorem, though affirmed by many authors, has never to his knowledge been demonstrated rigorously, and he offers, therefore, a demonstration which he believes to be secure against every objection. Only a little later in the same year Poincaré † proposed a geometric proof of a different sort, of which he says that the two principal steps will possibly seem almost evident. Yet there will perhaps be some interest, he continues, in having a demonstration which is secure against every objection. In 1895 Appell and Goursat ‡ refer in a footnote to the proofs of Simart and Poincaré, but rely for their establishment of the theorem upon another very interesting transformation previously used by Halphen § to secure a curve with ordinary singular points only. In 1896 Vessiot || states that many demonstrations of the theorem have been given, but that the modification of Appell and Goursat's proof which he is about to explain will not be without interest on account of its simplicity and complete rigor. In a footnote ¶ he says that if one accepts as evident the steps (*e*) and (*f*) of his demonstration, as Appell and Goursat have done at the analogous points of their proof, one may arrive at his conclusion more rapidly. But the explanations which he gives seem to him indispensable if one desires to establish these two points beyond all doubt.

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* *Étude sur les points singuliers des courbes planes algébriques*, appendix to Salmon's *Traité de Géométrie Analytique* (*Courbes Planes*), translated by O. Chemin, edition of 1903, pp. 627–31.

† *Traité d'Analyse*, vol. 2, 1893, p. 364.

It seems to me that these comments express considerable dissatisfaction and uncertainty with regard to the proofs of Theorem II which were at that time under discussion. The transformations suggested by Halphen are most interesting and plausible, and one would expect, as Picard implies, that their effectiveness can be conclusively established. I do not feel convinced as yet that this has been done. Simart's paper is so concise that I have so far been unable to verify all the

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* COMPTES RENDUS, vol. 116 (1893), p. 1047.

† COMPTES RENDUS, vol. 117 (1893), p. 18.

‡ *Théorie des Fonctions Algébriques*, 1895, p. 282.

§ JOURNAL DE MATHÉMATIQUES, loc. cit.

|| ANNALES DE TOULOUSE, vol. 10 (1896), p. D1.

¶ Page D6.

steps of his argument to my own satisfaction. Vessiot uses a combination of projective transformations with the simple transformation

$$\xi = x, \qquad \eta = \frac{dy}{dx} = -\frac{f_x}{f_y}.$$

One cannot in general by a preliminary projective transformation prevent this function η from having poles in the finite part of the Riemann surface of $f(x, y) = 0$. Each of these poles provides a branch which in the projective $x\eta$-plane passes through the intersection of the η-axis and the line at infinity. A singular point is therefore introduced at infinity on the projective curve which cannot be made to disappear by a subsequent projective transformation. It does not seem possible, therefore, to prove the geometric Theorem II by this method. On the other hand I am inclined to believe that Vessiot's method may afford a relatively simple proof of the function-theoretic Theorem II. By a projective transformation one should be able to bring it about that $\eta = dy/dx$ has distinct finite values at the infinite points of the Riemann surface, and at most simple poles over distinct values of x in the finite parts of the surface. These properties insure the existence of simple points only at the intersections of the function-theoretic curve with the rays $x = \infty$ and $y = \infty$ through the points A and B. The methods of Vessiot could then be used to secure at most ordinary double points at finite points (x, y). Some of the steps are, however, analytically delicate.

7. *The Bertini Group of Proofs of Theorem* II. A third group of proofs, this time of the geometric Theorem II, is due to Bertini.* In 1891 he was apparently unaware of Halphen's paper of 1884, for he states that in many works on geometry Theorem II is applied, but that as far as he knows no one has given an explicit demonstration. He proposes, therefore, one which appears to him to be simple and rigorous. It seems likely that the form which he devised was suggested by some studies of birational transformations of curves published by Clebsch and Lindemann in 1876.† In 1894 Bertini republished

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* Rivista di Matematica, vol. 1 (1891), p. 22.

† *Vorlesungen über Geometrie*, 1876, pp. 661 ff.

his proof in the MATHEMATISCHE ANNALEN,[*] and in a footnote explained that a geometric difficulty which Poincaré had avoided somewhat circuitously in 1893 had already been discussed by himself. Klein added a second footnote in which he suggested an interesting geometric transformation, equivalent to Bertini's, which he says had been communicated to him orally by Clebsch in 1869. In 1905, in the second edition of the second volume of his *Traité d'Analyse*,[†] Picard adopts Bertini's proof with slight modifications only. In 1906 Walker [‡] remodeled Bertini's method in detail in accordance with Klein's geometric suggestion. In Walker's introduction he refers to the proofs listed in this and the preceding groups, and remarks that they are written in very concise style and leave a great many minor points to the reader. In 1907 W. H. and G. Ch. Young [§] described a new condition on Bertini's transformation which must be added in order to make it entirely satisfactory. They were evidently unacquainted with Walker's thesis, in which this objection had been answered in a different way but had not been pointed out explicitly.

The great advantage of the form of Bertini's proof adopted by Walker lies in the fact that the various steps of the proof have geometric interpretations in ordinary spaces of two and three dimensions. It is consequently easier by this method to classify the properties which must be imposed upon Bertini's transformation, and to make sure that all of them have been suitably secured. The proof is in outline as follows. Through six points of a plane π, not on a conic and no three collinear, there pass four linearly independent cubic curves $f_i(x_1, x_2, x_3)$, $(i = 1, \cdots, 4)$. The equations

$$\rho y_i = f_i(x_1, x_2, x_3), \qquad (i = 1, \cdots, 4) \tag{5}$$

define a cubic surface F in the three-dimensional y-space, and the points of F are in one-to-one correspondence with those of the plane π with the exception that six of the straight lines on the surface correspond to the six base points. For each of these lines there is, however, a one-to-one correspondence between its points and the rays through the corresponding base point. Let K be an irreducible algebraic curve in the x-plane with ordinary singular points only, and let the base points of the transformation just described be selected so that one of them is at a singularity A of K but the others not on K. Then the curve K is transformed by equations (5) into a new curve K' on the surface F, and K' has singularities similar to those of K with the exception that one of them, namely A, is dispersed into ordinary points. If a suitable center of projection O is now taken on the surface F, the curve K' can be projected into a plane curve K'' which has possibly new ordinary double points in place of A, but whose other singularities are the same in number and in type as those of K. By a succession of such transformations all of the singular points of K may therefore be replaced by ordinary double points.

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[*] Vol. 44 (1894), p. 158.

[†] Page 408.

[‡] *On the resolution of higher singularities of algebraic curves into ordinary nodes*, Dissertation, Chicago, 1906.

[§] TORINO ATTI, vol. 42 (1907), pp. 82–6.

The difficult part of the demonstration is the proof that it is possible to choose properly the center of projection O. If O is selected on F but distinct from K' and the straight lines of F, and distinct from the cones generated by K' at its own singular points, then the projection of K' from O defines a curve K'' in one-to-one correspondence with K' except for a finite number of double points of K'' where the correspondence is one-to-two. The choice of O in this manner on the cubic surface F effectively prevents any ray through O from being a trisecant of K'. If O is not on the developable of the tangents to K', then K'' will have no cusps. If O is not on the tangent planes to F at the multiple points of K', and not on the cones generated by K' at its own singular points, then each multiple point of K' will project into an ordinary multiple point of the same character for K''. Finally if O is distinct from the ruled surface whose generators are lines joining points of K' with coplanar tangents, then the only double points introduced for K'' will be ordinary double points. In order to make the proof complete it must be shown that each of the loci just mentioned exists and can be avoided in the selection of the point O. Walker has not only done this but he has also discussed in detail the properties of the cubic transformation from the plane π to the surface F. His thesis fills all told about fifty pages, a decided contrast to Bertini's original outline of slightly more than one page. I have been satisfied, by Walker's reasoning or by arguments of my own which seemed to me more convenient, that all of his steps are well justified. It would be highly desirable to have the proof shortened and simplified without sacrificing clearness or accuracy, and I am inclined to think that it could be done.

8. *Other Geometric Proofs of Theorem* II. There is a fourth and final principal group of proofs of Theorem II, more geometric in character, with which I am relatively much less familiar.* The plan underlying most of them is to transform the given plane algebraic curve into a twisted curve in a higher space devoid of singularities, and then to project this non-singular curve upon a two-dimensional plane in such a way that only ordinary double points remain. Intuitively the possibility of these two steps, as Poincaré says, does seem almost evident, and it is perhaps for this reason that in most of the geometric proofs the final projection is hardly mentioned. It seems to me, however, that this projection is a really difficult part of the proof. Poincaré's proof, for example, is very condensed, but all of his steps can be verified without serious difficulty up to his discussion of the projection of the non-singular curve in higher space into a non-singular curve in three-space. At this point his reasoning seems to me much too concise, and even questionable. In Walker's form of Bertini's proof the projection of the space curve K' on the surface F into a plane curve K'' is the most difficult step. My doubts about the ease of making the projection have been strengthened by

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* Veronese, MATHEMATISCHE ANNALEN, vol. 19 (1881), p. 213; Poincaré, loc. cit.; del Pezzo, NAPOLI RENDICONTI, (2), vol. 7 (1893), p. 15; Vessiot, BULLETIN DE LA SOCIÉTÉ MATHÉMATIQUE DE FRANCE, vol. 22 (1894), p. 208; Pieri, RIVISTA DI MATEMATICA, vol. 4 (1894), p. 40; Segre, ANNALI DI MATEMATICA, (2), vol. 25 (1897), p. 43; Severi, *Lezioni di Geometria Algebrica*, 1908, p. 172; Del Re, MODENA MEMORIE, (2), 10 (1894), p. 447, footnote; Jamet, Nouvelles Annales, (3), vol. 19 (1900), p. 506.

some further interesting evidence. After I had made the proof of the geometric Theorem II, following Kronecker, I read Severi's geometric proof and was surprised to find a close analogy between my own argument and his. This was a result of the relationship between the theory of rational functions of x, y with prescribed poles on the Riemann surface of $f(x, y) = 0$ and the geometric theory of the groups of points intersected on the curve $f(x, y) = 0$ by linear families of curves. It is these linear families which Severi uses to get his transformation from the plane curve to a non-singular curve in higher space. But Severi relegates the question of the projection of his space curve upon a plane, so as to leave only ordinary double points, to a short footnote containing only some remarks on how to avoid trisecants, whereas in the Kronecker theory the equivalent steps form the most delicate part of the proof. For the present, then, I am discontented, to say the least, when I find a geometric proof which does not explain in detail the projection mentioned. The transformation from the original plane curve to the non-singular space curve is relatively easy and can be done in a number of ways.

9. *Origin of Theorem* II. You will understand, I think, from what has been said above, that it is not easy to designate with certainty the origin of Theorem II. Halphen in 1884 ascribed the theorem to Noether, and he has since been imitated by a number of other writers, probably on account of the close relationship between Theorem II and Noether's Theorem I. On the basis of published papers it would be difficult to establish this origin. Klein, after stating the theorem without proof in the text of his *Riemannsche Flächen*,* inquires in a footnote "Wo ist der Satz zum ersten Male gedruckt?", and states that he himself first heard it orally from Kronecker in the autumn of 1869. Hensel and Landsberg also ascribe the theorem to Kronecker. If we rely on printed memoirs, however, one must admit that Kronecker's paper, first published in 1881, was devoted to the analysis of the discriminant of an algebraic function and was essentially analytic in character. His

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* 1892, p. 245.

reduction has indeed an important bearing upon Theorem II, but it affords only a partial proof. You will remember that Klein, in the footnote which he added to Bertini's ANNALEN paper of 1894, states that the geometric transformation there described was communicated to him orally by Clebsch in 1869. Brill and Noether* further publish an extract from a letter of Klein to Noether dated December 17, 1869, in which Klein describes two results attained, respectively, by Kronecker and Clebsch. The first is the transformation of an arbitrary algebraic plane curve into another with no singularities except a single ordinary singular point; and the second is the transformation of a curve with a single ordinary singular point into one with at most ordinary double points by the geometric process which Klein ascribes to Clebsch in his footnote to Bertini's paper. Klein further adds in his letter that he believes Clebsch's method can be extended so as to reduce any number of arbitrary singularities to ordinary double points. Brill and Noether contend, and it seems to me with justice, that these assertions would be more effective as the basis of a claim for priority in the proof of the theorem if they had been afterwards substantiated in print. In 1897 Segre,† speaking of the transformation of a plane curve into one in higher space with no singularities, says that solutions of the problem are contained implicitly in a paper by Brill and Noether of 1874,‡ and more explicitly in Veronese's paper of 1881 previously cited. As I said above, however, the first explicit published statement of the theorem with proof was probably that of Halphen in 1884. On the whole, I am inclined to agree with the conclusion implied by an above quoted statement of Bertini in 1891. The theorem seems to have gradually established itself in the consciousness of mathematicians during the decades from 1860 on, and an assignment of it to a definite origin would probably be incorrect. It is without doubt true that many a theorem now confidently attributed to a particular mathematician would upon closer examination prove rather to be, as this one seems to be, the product of a gradual evolution in the minds of many individuals.

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* JAHRESBERICHT, loc. cit., p. 371.

† Loc. cit., p. 43.

‡ MATHEMATISCHE ANNALEN, vol. 7 (1874), p. 269.

It seems to me that there is still a need for concise and clear proofs of the function-theoretic and the geometric Theorems II. The Hensel and Landsberg proof of the former, and Walker's proof of the latter following Bertini and Klein, are the only published ones with whose details I am at the present time personally well content. There are objections to both of them. It would be very difficult and inconvenient to analyze and understand the former unless one had read many of the preceding pages of Hensel and Landsberg's book, and possibly also Kronecker's original memoir. Walker's proof has the sort of geometric background which should enable one to decide definitely whether or not all essential details have been thought of and suitably discussed. But it is very long, and, having been published privately, it has not been subjected to the close scrutiny of mathematicians in various centers which seems to me desirable. The last objection applies also to my own proof of which I spoke above. So far as I know there is at present only one person who has definitely, though somewhat timidly, expressed his approval of this effort of mine, and his identity I must leave you to surmise. In conclusion I have only one further hope to express, namely, that no new proof will appear in the very condensed style which makes it almost impossible to decide whether the theorem has really been established or not. There are a number of these already, and the accumulation of others would only add to the difficulties of every earnest searcher for the truth in this domain. It seems to me that brevity at the expense of clearness, in a situation of this sort, should not be mistaken for simplicity.

THE UNIVERSITY OF CHICAGO

SHORTER NOTICES

James Stirling: A Sketch of his Life and Works, along with his Scientific Correspondence. By Charles Tweedie. Oxford, Clarendon Press, 1922. x + 213 pp.

This book is an interesting and valuable contribution to the history of mathematics, giving as it does a considerable amount of new information about the work of the early successors of Newton. One of the ablest of these was James Stirling, and thus his hitherto unpublished mathematical correspondence contains much of interest.

The book is divided into three parts, "Life," 22 pages, "Works," 26 pages, and "Correspondence," 160 pages. The first part adds some interesting details to the rather meagre stock of information available concerning Stirling's life. He was one of the non-juror students at Oxford, and as such under suspicion at the time of the accession of George I, when there were riots and a revolution in favor of the restoration of the Stuarts seemed imminent. In fact, because of his Jacobite leanings, Stirling was deprived of a scholarship which he had previously held; but the usual account * that he was expelled from the University and "fled to Italy" is incorrect, as the author brings out the fact that Stirling only left Oxford in 1717, to go to Venice on the offer of a mathematical professorship there. This was the year in which he published (in Oxford) his book on cubic curves. The offer in Venice was eventually declined, on account of the religious conditions attached to it. He remained in Italy, however, till about 1724, when he located in London. He was a friend of Newton and of Maclaurin, and was highly esteemed by other contemporary mathematicians, particularly Machin, DeMoivre, and Cramer. The latter part of his life (from the age of 43 on) he devoted to commercial work, and thus his mathematical productivity came to an end just at the time when he was in a position to have added much to the development of the subject.

Stirling's chief works are *Lineae Tertii Ordinis* (1717) and *Methodus Differentialis* (1730). These are well described in the second part of the book under review. The *Methodus Differentialis* contains most of Stirling's own contributions to mathematics; these relate to the summation of series, to interpolation, and to approximation formulas, the most famous of the last being "Stirling's Formula,"

$$n! = \sqrt{2n\pi}\cdot n^n\cdot e^{-n}.$$

He finds the exterpolated value of the term preceding the first in the series of factorials,

$$1,\ 1,\ 2,\ 6,\ 24,\ 120,\ \text{etc.}$$

to be 1.7724538502, and says that this is equal to $\sqrt{\pi}$. He also introduces the Beta Function as an integral, for the interpolation of

$$a,\quad \frac{r}{p}\,a,\quad \frac{r(r+1)}{p(p+1)}\,a,\quad \cdots,$$

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* *Dictionary of National Biography*; Cantor, *Geschichte der Mathematik*, vol. 3, p. 372.

and obtains a result equivalent to

$$\frac{B(p+n',q)}{B(p'q)}=\frac{p(p+1)\cdots(p+n-1)}{(p+q)(p+q+1)\cdots(p+q+n-1)}.$$

The third part of the book is the most important, both in volume and in the additional light which it throws upon Stirling's work in its relation to his contemporaries. Thus there are 11 letters from Maclaurin to Stirling, 10 from Cramer, 3 from Nicholas Bernoulli (1687–1750), the nephew of John Bernoulli, 2 from Machin, and one, perhaps the most valuable in the entire collection, from Euler. There are 9 letters written by Stirling (4 to Maclaurin, 1 to Cramer, 1 to N. Bernoulli, 1 to Castel, 1 to Bradley, and 1 to Euler). All the letters to Stirling testify to the high regard in which the writer held him, and to the influence which his writings exerted upon contemporary mathematics. The letter from Euler mentioned above, contains a rich fund of information about his work in series, including $\Sigma_{n=0}^{\infty}1/n^k$, $\Sigma_{n=0}^{\infty}1/(2n+1)^k$, and others. A curious infinite product which he gives is perhaps not generally known:

$$\frac{3\cdot5\cdot7\cdot11\cdot13\cdot17\cdot19\cdot23\cdot29\cdots}{4\cdot4\cdot8\cdot12\cdot12\cdot16\cdot20\cdot24\cdot28\cdots},$$

where the factors of the numerator are the successive prime numbers, and each factor of the denominator is the integer of form $4k$ nearest to the corresponding factor in the numerator. Euler says the limit of this fraction is $\pi/4$, but does not give any hint as to his method of proof.

Mr. Tweedie has done a valuable piece of work, which will be found useful not only as a contribution to the history of mathematics, but also as giving in very readable form a source of information as to one of the founders of the modern theory of finite differences.

R. B. McClenon

The Fourth Dimension and the Bible. By W. A. Granville. Boston, Richard G. Badger, 1922. 9 + 119 pp.

The author of this little book points out in his preface that there is little hope of building a system of theology on a foundation either of philosophy or of the physical sciences, owing to the fact that neither of these branches of knowledge present results of a sufficiently authoritative character. He calls attention to the fact that the very foundations of the physical sciences themselves are at the present time open to the most serious challenge. Mathematics is the only existing body of knowledge which may be regarded as entirely authoritative. The author accordingly hopes "to throw some light, even though very dim, on some of the questions connected with our Christian beliefs." His chief aim is "to point out the remarkable agreement which exists between numerous Bible passages and some of the concepts which follow naturally from the mathematical hypothesis of higher spaces." As a popular introduction to some of the fundamental notions of four-dimensional space, the book is well done, although it seems to the reviewer that the author has rather over-emphasized the notion of reasoning by analogy. As to the value of his contribution to theology, let others speak.

J. W. Young

Ballistik. By Dr. Theodor Vahlen. Berlin, Vereinigung wissenschaftlicher Verleger, 1922. xi + 226 pp.

This book is an outgrowth of interest and investigation that were stimulated in the subject of ballistics during the World War. The author, who is professor of pure and applied mathematics in the University of Greifswald, served as an artillery officer, and during that period contributed a number of articles on ballistics to scientific and to military journals. These articles are referred to in the beginning of the book where a rather extensive bibliography on the subject is given, from the work of John Bernoulli in 1719 to the works of Cranz and of Lorenz in 1917.

The reason for writing the book is that there is no book in German literature which presents the mathematical developments of the subject in detail. Cranz has deliberately avoided these in his compendious work which is indispensable to the practical ballistician because of its useful tables and numerous illustrative examples. As no real progress in the science of ballistics is possible without a very close coordination between the theoretical and the practical, the author aims to supply a definite demand by presenting the subject from the standpoint of the mathematician.

There are four divisions of the subject: exterior ballistics, the behavior of the projectile after the powder gas has exerted its entire accelerating effect; interior ballistics, the behavior of the projectile until it begins to leave the muzzle; transition (Übergangs) ballistics which embraces the interval between exterior and interior; and, *Endballistik* which includes the functioning of shells, recoil, etc. The subject is divided further into old ballistics which rests upon the assumptions that the earth is flat, motionless, that gravity, both in magnitude and direction, and the density of the air (upon which the resistance of the air depends) are constant throughout the trajectory, and modern ballistics in which greater ranges and altitudes are encountered and in which the former assumptions are no longer adequate. The claim is made that both transition ballistics and, what I have called *modern* ballistics are treated systematically for the first time in this book. The author seems disposed to name the latter *universal* (that is, as applying to the universe instead of to the earth only) ballistics but refrains from doing so in that sense because the motion of a meteor belongs to the field of celestial mechanics and cannot be regarded a problem of what he calls "kosmische" ballistics.

Under the head of modern ballistics he gives just about what has been developed in this country during the same period, both in regard to the trajectory itself and in regard to small corrections. If there is any difference, his mathematics is more elementary. The coordinates of a point on the trajectory are expressed as power series in t (the time); in this country the coordinates of a point on the trajectory are expressed as functions of velocity, acceleration, and a quantity E which depends upon velocity, altitude, and the ballistic coefficient. The two methods are theoretically the same if we admit that our x and y may be expressed as power series in t. If different results are obtained by the two methods, it would be due to inaccuracies in tabulated physical data.

As the author has so strongly stressed the necessity of developing the mathematical and the empirical aspects of the science side-by-side as a condition precedent to making any permanent progress, he is open to criticism on exactly those grounds in his treatment of correcting for curvature of the earth. To show that this correction is of practical value he takes the extraordinarily long range of 112.8 km. (about 70 miles) and points out that in this case the correction would be 1 km. "Surely such a quantity is not to be disregarded." But does it not seem so very large only because we have so little definite knowledge in regard to the probable error of the gun fired at such a range? Would it not have been more interesting to the average reader to know how the correction for a range of 8 or 10 miles compares with the mean deviation of a series of rounds fired at these more usual ranges? Also it might be well to include a numerical example taking the gun and the target at different heights above sea-level, especially since this could be done by using the formula which is given. The same sort of criticism might be made in regard to the corrections for rain and for an increase in the temperature of the gun due to firing. Experience in this country would seem to indicate that rain would have to be further classified into mist, heavy rain, etc.

In interior ballistics the author follows Sarru and Charbonnier. The three principal formulas are as follows: (I) connecting pressure and density of loading; (II) connecting the linear velocity of the burning of the powder with density and pressure; (III) the work done in the adiabatic expansion of a gas from a finite to an infinite volume. Other familiar formulas derived in this section are those relating to variation of pressure in the barrel, the maximum pressure, and the muzzle velocity.

Although the author emphasizes the importance of transition ballistics and the fact that he presents the subject systematically for the first time, he adds nothing to the subject which is not already very familiar to those interested in the subject. The projectile does not attain its maximum velocity at the muzzle but it is accelerated by the powder gas for some little time afterward. It is also during this interval that the projectile is given a yaw the effect of which, both upon range and upon deflection, is much greater than was formerly supposed. This is one of the problems that remains to be solved and it constitutes one of the most interesting and most advanced fields of investigation in ballistics. The only contribution is a definite name for this important division of the subject.

The later chapters contain a mathematical discussion of probability as related to problems in artillery and suggestions regarding indirect fire.

The author has been successful in making his book mathematical, which was one of his aims. In the reviewer's opinion he has not developed the subject as if the mathematical problems were suggested by observations made in practical experimentation. He pointed out the necessity for this and lamented the fact that it had not been done consistently in the past. The book will be of considerable interest to the mathematician who wishes to acquaint himself with the subject of ballistics; it will be of little value to the practical artillery officer or to the investigator in practical ballistics.

J. E. Rowe

NOTES

During the Summer Meeting of the Society (see this BULLETIN, vol. 29 (1923), p. 141), a joint session will be held with the Mathematical Association of America on September 6. It is expected that Professor L. E. Dickson will address this session on behalf of the Society on the topic *Algebras and their arithmetics.*

The forty-second regular session of the San Francisco Section of the Society will be held at the University of Southern California, in Los Angeles on September 17–20, 1923, in connection with the meeting of the Pacific Division of the American Association for the Advancement of Science. This announcement cancels the previously reported date and place of meeting (see this BULLETIN, vol. 29 (1923), p. 10). Abstracts should be in the hands of the Secretary of the Section, Professor B. A. Bernstein, not later than July 26.

In September, 1920, the delegates of the American Section of the International Mathematical Union extended an invitation to the Union to hold a Mathematical Congress in New York or vicinity in 1924, and the invitation was accepted at the meeting of the Union held in Strassbourg. Since that time the Section has found it not feasible to carry out in 1924 the plan originally proposed and has been obliged to withdraw its invitation. This action was taken in order that the Union should be free to consider other invitations which might be received for the Congress.

At the meeting of the Division of Physical Sciences of the National Research Council in Washington on April 22, the following officers were elected for the year beginning June, 1923: Chairman, Oswald Veblen; Secretary, W. E. Tisdale; Executive Committee, Oswald Veblen (chairman), William Duane, E. B. Frost, Theodore Lyman, W. F. G. Swann. The following persons were elected members-at-large: Bergen Davis, S. J. Barnett, Leigh Page, F. K. Richtmyer. New members representing scientific societies were announced as follows: for the American Mathematical Society, E. V. Huntington; for the American Astronomical Society, E. W. Brown; for the American Physical Society, H. M. Randall and Paul D. Foote; for the Mathematical Association of America, H. L. Rietz.

The first award of the Bôcher Prize of this Society (see this BULLETIN, vol. 28, pp. 42–44 and p. 378) will be made at the time of the annual meeting of the Society in New York, December 27 and 28, 1923.

The members of this Society will be glad to hear that, thanks to the generosity of twenty persons, the Expense Fund estimated necessary to cover the cost of raising the proposed endowment has been completely subscribed. It is believed that this will more than take care of all overhead charges, and every cent contributed in future will go directly to the purposes for which the endowment is planned.

The ANNALI DI MATEMATICA will begin its fourth series with the next volume, the first number of which is now in press. It will be published by Nicolo Zanichelli, at Bologna, under the present board of editors.

The price for subscribers outside of Italy has been fixed at 50 French francs per volume, but subscriptions received prior to March 1, 1923, at the former rate of 40 lire, will be honored for the first volume. Owing to the generous response to the recent appeal, no further cash contributions are now needed, but more subscriptions are desired and invited. These should be sent either to the publisher in Bologna, or to Professor Virgil Snyder, Ithaca, New York.

A continuation of Professor L. P. Eisenhart's *Differential Geometry* has just appeared under the title *Transformation of Surfaces*. This is the first book to be published with the cooperation of the National Research Council in connection with its new plan of a revolving book fund for the publication of advanced treatises in mathematics. Professor Eisenhart has given his book to help in this plan, in the sense that all profits from the sale of the book go to increase the fund. The Princeton University Press, which is acting technically as the publisher of the book, has generously offered to exercise its functions without charge. The book contains 380 pages, and its price is four dollars. It is to the advantage of the fund to have purchases made directly through the Princeton University Press.

The following advanced courses in mathematics are announced at the University of Chicago for the Summer Quarter, 1923 (June 18 to August 31). All courses meet four times a week. By Professor L. E. Dickson: Differential equations from the standpoint of Lie; Arithmetic and algebra of hypercomplex numbers.—By Professor F. R. Moulton: Analytic mechanics; Theory of functions of a complex variable.—By Professor A. C. Lunn: Theory of relativity; Theory of sound.—By Professor L. S. Mordell: Theory of definite integrals; Analytic theory of numbers.—By Professor E. J. Wilczynski: Theory of linear differential equations.—By Professor J. W. A. Young: Selected topics in mathematics.—By Professor E. J. Moulton: Advanced calculus.—By Dr. Mayme I. Logsdon: Higher plane curves.—By Professor A. F. Carpenter: Solid analytic geometry.

Mr. Dan Brouse, a graduate of Purdue University, has been appointed assistant professor of mathematics at Baker University, Baldwin, Kansas.

At Kansas State Agricultural College, Mr. W. H. Rowe, Mr. W. C. Janes, and Miss Thirza A. Mossman have been appointed instructors in mathematics.

At Iowa State College, Messrs. F. A. Brandner, H. P. Doole, P. O. Robinson, and Henry Giese have been appointed instructors in mathematics.

Mr. Macon Reed, of Columbia University, has been appointed head of the department of mathematics at Hampden-Sidney College.

Miss Rose B. Wood, a graduate of Barnard College, has been appointed professor of mathematics at Greenville (S. C.) Woman's College.

At the University of South Carolina, Mr. M. A. Hill and Mr. C. C. Edwards have been appointed instructors in mathematics.

Mr. R. G. Demaree, of the University of Chicago, has been appointed head of the department of mathematics and physics at Kentucky Wesleyan College.

NEW PUBLICATIONS

I. PURE MATHEMATICS

APPELL (P.) et LACOUR (E.). Principes de la théorie des fonctions elliptiques et applications. 2e édition. Paris, Gauthier-Villars, 1922.

BAKER (H. F.). Principles of geometry. Volume 2: Plane geometry, conics, circles, non-euclidean geometry. Cambridge, University Press, 1922. 16 + 244 pp.

BELTRAMI (E.). Opere matematiche. Tomo 4 ed ultimo. Milano, Hoepli, 1920. 4to. 554 pp.

BRADLEY (F. H.). The principles of logic. 2d edition. Volumes 1–2. London, Oxford University Press, 1922. 28 + 388 + 350 pp.

FUBINI (G.). Lezioni di analisi matematica. 4a edizione, interamente rifusa. Torino, Società Tipografico-Editrice Nazionale, 1920. 8 + 470 pp.

GOURSAT (E.). Cours d'analyse mathématique. 3e édition. Tome 3. Paris, Gauthier-Villars, 1923.

HAAS (A.). Vektoranalysis in ihren Grundzügen und wichtigsten physikalischen Anwendungen. Berlin, Vereinigung wissenschaftlicher Verleger, 1922. 6 + 149 pp.

KRAÏTCHIK (M.). Décomposition de $a^n \pm b^n$ en facteurs dans le cas où nab est un carré parfait avec une table des décompositions numériques pour toutes les valeurs de a et b inférieures à 100. Paris, Gauthier-Villars, 1922.

KUMM (E.). Das Potential gewisser homogener Rotationsovaloide. (Diss., Halle-Wittenberg.) Halle, Buchdruckerei Hohmann, 1922. 45 pp.

LACOUR (E.). See APPELL (P.).

LEHNEN (M.). Eine Theorie der Raumkurven 3. 0. auf Grundlage der Invariantentheorie. (Diss., Bonn.) Bonn, Druck von T. Wurm, 1921. 48 pp.

MUICA (I.). Quelques observations élémentaires sur les nombres entiers. Théorème de Fermat. 2e édition. Bucarest, Viata Românească, 1922. 12 pp.

NEVILLE (E. H.). Prolegomena to analytical geometry in anisotropic euclidean space of three dimensions. Cambridge, University Press, 1922. 22 + 368 pp.

POPPOVICH (N. M.). Die Lehre vom diskreten Raum in der neueren Philosophie. Wien und Leipzig, Braumüller, 1922. 2 + 89 pp.

SEN GUPTA (J.). The fundamentals. Volume 1. Calcutta, Kar Majumder and Company, 1921. 8vo. 85 pp.

WEISNER (L.). Groups whose maximal cyclic subgroups are independent. (Diss., Columbia.) New York, 1923. 17 pp.

WIELEITNER (H.). Geschichte der Mathematik. I: Von ältesten Zeiten bis zur Wende des 17tes Jahrhunderts. (Sammlung Göschen.) Berlin, Vereinigung wissenschaftlicher Verleger, 1922. 136 pp.

II. APPLIED MATHEMATICS

Adler (F.). Ortzeit, Systemzeit, Zonenzeit und das ausgezeichnete Bezugssystem der Elektrodynamik. Eine Untersuchung über die Lorentzsche und die Einsteinsche Kinematik. Wien, Wiener Volksbuchhandlung Ignaz Brand, 1920. 8vo. 16 + 237 pp.

Bhagwat (V. K.). Calculations in organic chemistry. Bombay, S. Govind and Company, 1921. 11 + 138 pp.

Bjerknes (V.). Untersuchungen über elektrische Resonanz. Leipzig, Barth, 1922. 32 + 129 pp.

Bohr (N.). The theory of spectra and atomic constitution. Three essays. Cambridge, University Press, 1922. 10 + 126 pp.

Bouasse (H.). Hydrostatique. Manomètres, baromètres, pompes. Equilibre des corps flottants. Paris, Delagrave, 1923. 24 + 480 pp.

Bridgman (P. W.). Dimensional analysis. New Haven, Yale University Press, 1922. 5 + 112 pp.

Chapman (R. W.). The elements of astronomy for surveyors. 2d edition, revised. London, Griffin, 1922. 261 pp.

Drosne (P.). Le beau, le mauvais temps et la relativité. Paris, Doin, 1922. 18mo. 68 pp.

Eucken (A.). Grundriss der physikalischen Chemie. Leipzig, Akademische Verlagsgesellschaft, 1922. 8 + 492 pp.

Fischer (L.). Die Wärme ein Gas! Eine neue Theorie der Wärme und der übrigen feinen Stoffe. Leipzig, H. A. Ludwig Degener, 1922. 61 pp.

Gröber (H.). Die Grundgesetze der Wärmeleitung und des Wärmeüberganges. Berlin, Springer, 1921. 271 pp.

Hanocq (C.). Les pompes centrifuges à haute pression. Paris et Liège, Béranger, 1921. 8vo. 160 pp.

Institut international de Physique Solvay. Atomes et électrons. Rapports et discussions du Conseil de Physique tenu à Bruxelles du 1er au 6 avril 1921. Paris, Gauthier-Villars, 1921. 8 + 273 pp.

Kriemler (C.). Technische Mechanik. Ein Lehrbuch der Statik und Dynamik starrer und nachgiebiger Körper. 2te, vermehrte und verbesserte Auflage. Stuttgart, 1920.

Lebert (E.). Electricité, magnétisme, gravitation, mécanique, théorie de la chaleur. Vannes, Lafolye frères, 1922.

Loring (F. H.). Definition of the æther. London, H. O. Lloyd, 1922. 16 pp.

Müller (A.). Der Gegenstand der Mathematik mit besonderer Beziehung auf die Relativitätstheorie. Braunschweig, Vieweg, 1922. 8 + 94 pp.

Müller (E.). Technische Uebungsaufgaben für darstellende Geometrie. Heft 5. Leipzig und Wien, Deuticke, 1921.

Newest (T.). Gegen Einstein: Die Erfahrung im Weltall. Einige Weltprobleme. Allgemeinverständliche Abhandlungen. 2te Auflage. Wien, Frisch, 1921. 8vo. 95 pp.

Pecheux (H.). Traité d'électricité industrielle. Paris, Delagrave, 1922. 2 volumes. 834 + 376 pp.

Petrovitch (M.). Mécanismes communes aux phénomènes disparates. Paris, Alcan, 1921. 8vo. 279 pp.

Pflüger (A.). Das Einsteinsche Relativitätsprinzip gemeinverständlich dargestellt. 3te Auflage. Bonn, 1920.

Podevyn (P. A.). Traité de mécanique, à l'usage des ouvriers, dessinateurs, et techniciens-mécaniciens. Paris, Librairie Desforges, 1922. 16mo. 169 pp.

Rinne (F.). Das feinbauliche Wesen der Materie nach dem Vorbilde der Kristalle. 3te Auflage. Berlin, Gebrüder Borntraeger, 1922. 8 + 168 pp.

Ripke-Kühn (L.). Kant contra Einstein. Erfurt, Keyserschen Buchhandlung, 1920.

Schau (A.). Festigkeitslehre. 2te Auflage. (Aus Natur und Geisteswelt.) Leipzig, Teubner, 1921. 111 pp.

Sepp (H.). Stoff und Kraft in Verbindung mit Raum und Zeit, nach den neuesten Forschungen über Atomzerfall sowie nach der Quanten- und der Einstein'schen Relativitätstheorie. Berlin-Steglitz, Heimatverlag M. Hiemesch, 1921. 8vo. 32 pp.

Solvay. See Institut.

Tumlirz (O.). Die Zustandsausgleichung des flüssigen und dampfförmigen Wassers. Wien, Hölder, 1921. 8vo.

Verschoyle (W. D.). The evolution of atoms and isotopes. London, Keliher, 1922. 40 pp.

Wantoch (R.). Die Einsteinsche Relativitätstheorie. Kurze, für jedermann verständliche Besprechung. Wien, Hölder, 1921. 8vo. 15 pp.

Weyl (H.). Raum, Zeit, Materie. Vorlesungen über allgemeine Relativitätstheorie. 5te, umgearbeitete Auflage. Berlin, Springer, 1923. 8 + 338 pp.

THE APRIL MEETING OF THE SAN FRANCISCO SECTION

The forty-first regular meeting of the San Francisco Section of the American Mathematical Society was held at Stanford University on April 7. Professor Cajori presided. The total attendance was twenty-three, including the following seventeen members of the Society:

Alderton, Allardice, Bernstein, Blichfeldt, Buck, Cajori, A. F. Carpenter, Growe, Hoskins, Lehmer, Levy, Libby, McEwen, F. R. Morris, Smail, Pauline Sperry, A. R. Williams.

The Secretary read a communication from Secretary Richardson asking the members of the Society to cooperate with the Endowment Committee, headed by Professor Coolidge, in the forthcoming campaign for funds for the Society.

In response to a request by a committee from the University of Washington for more active participation in the work of the Section by members of the Society living in and about Seattle, it was decided to hold a special meeting of the Section at Seattle in December, 1923, with Professor Carpenter acting as Secretary.

It was decided to hold the next regular meeting of the Section at Los Angeles, September 17–20, in connection with the meeting of the Pacific Division of the American Association for the Advancement of Science.

Titles and abstracts of papers read at this meeting follow. The papers of Professors Bell and Neikirk were read by title.

1. Professor E. T. Bell: *Euler algebra.*

This name is given to the algebra of certain infinite processes fundamental in the study of the multiplicative properties of numbers. Two other algebras, called here the Cauchy and Dirichlet varieties, are also investigated. The first is that of power series, the second that of Dirichlet series. Euler algebra is a species of resultant between these two. The paper is to appear in the TRANSACTIONS OF THIS SOCIETY.

2. Professor E. T. Bell: *A class of numbers connected with partitions.*

The theory of partitions and that of the representation of an integer as a sum of squares originating in the elliptic modular functions must be closely interconnected. The connections depend upon eight new systems of integers. These integers are functions of two parameters. For values $\ngtr 6$ of the parameters the numbers are connected with the class numbers of binary quadratic forms of a negative determinant. The paper will be published in the AMERICAN JOURNAL.

3. Professor E. T. Bell: *Square-partition congruences.*

This paper will appear in full in an early issue of this BULLETIN.

4. Professor E. T. Bell: *Theta functions and arithmetic.*

Any relation whatever between theta functions of $p > 1$ arguments implies and is implied by a corresponding relation between the solutions of a set of $\frac{1}{2}p(p+1)$ indeterminate equations of the following sort: each equation in the set is restricted only in that its coefficients must be (arbitrary) integers; the equations are not necessarily homogeneous; the degrees of the several equations in the set may be any $\frac{1}{2}p(p+1)$ positive integers, and likewise for the numbers of indeterminates. The excepted case $p = 1$ is treated in a forthcoming paper in the TRANSACTIONS OF THIS SOCIETY.

5. Professor E. T. Bell: *Note on total representations as sums of squares.*

If $O(r)$, $E(r)$ denote respectively the total numbers of representations of r as the sum of an odd, an even number of squares with roots different from zero, $O(2n+1) > E(2n+1)$, $E(2n) > O(2n)$ for all positive integers n.

6. Professor B. A. Bernstein: *Complete set of representations of two-element algebras.*

The author gives simple arithmetic representations of all two-element abstract algebras, with some applications to fundamental problems connected with postulate-sets.

7. Professor H. F. Blichfeldt: *Note on quadratic forms.* Preliminary report.

Given a positive quadratic form f of determinant D in n variables. Integers, not all zero, may be substituted for the variables in this form such that the numerical value of f is $\leqq \gamma_n \mid D \mid^{1/n}$, where γ_n is a function of n only. For $n < 6$, the lowest permissible values of γ_n have been found; γ_6 was proved $< \sqrt{3}$ by Korkine and Zolatareff in MATHEMATISCHE ANNALEN, vol. 6 (1873), p. 378. The present author proves that

$$\gamma_6 < \{8\sqrt[4]{72}\}^{1/6} \doteq 1.690\cdots.$$

8. Professor Florian Cajori: *Varieties of minus signs.*

It is shown that besides the sign — there was introduced, in 1525, ÷ as a minus sign. This maintained its place in a few mathematical books until the latter part of the nineteenth century. Other minus signs found in books are ∸, --, ---, ∼.

9. Professor Florian Cajori: *The evolution of the modern exponential notation.*

The advance toward our modern exponential notation made by the forerunners of Descartes, namely by Chuquet, Bombelli, Digges, Cataldi, Stevin, Romanus, Girard, Bürgi, Reymer, Kepler, Schoner, Hérigone and Hume, is traced with fullness of detail; also the spread of the cartesian symbols and of Newton's extension of it.

10. Professor A. F. Carpenter: *Flecnodal properties of a ruled surface.*

This paper has to do with certain quadrics connected with the flecnode and complex curves of a ruled surface, together with four related space cubics. The theorems deduced involve projective properties of these curves and surfaces, the method of attack being that of the projective differential geometry. The paper will appear in the TÔHOKU JOURNAL.

11. Professor D. N. Lehmer: *On congruences connected with magic squares with odd number of cells.*

The familiar *uniform step* method of constructing odd magic squares is based on certain interesting congruences which seem not to have been studied. One can tell by means of these congruences whether a square constructed with a given *step* and a given *breakstep* will be magic, and whether it also will be *diabolic* and *symmetric*.

12. Professor L. I. Neikirk: *Some non-associative linear algebras.*

The units of these algebras are transformations in a Galois field, and are finite in number. Two kinds of third and higher powers of units may exist, according to the way the factors are associated, and may be designated as right and left handed powers. Division is not always possible; a unit may not have either a right hand or a left hand inverse.

13. Professor L. L. Smail: *Some theorems on uniform convergence of infinite products.*

Using theorems by Bôcher and Birkhoff on the uniform convergence of $\sum u_n$ and $\sum |u_n|$ (ANNALS OF MATHEMATICS, vol. 4, p. 159, vol. 6, p. 90), a set of theorems are developed giving relations between $\Pi(1 + u_n)$, $\Pi(1 + |u_n|)$, $\sum u_n$, and $\sum |u_n|$.

14. Professor L. L. Smail: *Note on derivatives of a vector product.*

An interesting and simple formula for the nth derivative of the cross- or vector product of two vectors is obtained.

15. Professor L. L. Smail: *Report on a synopsis of the theory of summable infinite processes.*

The author reports that he has in preparation a book on *Synopsis of the Theory of Summable Infinite Processes*, which is to cover exhaustively the field of summable series and other processes, by presenting without proof all the important results, definitions and theorems thus far obtained. This would do for the subject of summability what Dickson's *History of the Theory of Numbers* has done for that subject.

16. Dr. Pauline Sperry: *On real elements in imaginary linear spaces.*

In this paper the author shows that all real points in an imaginary linear k-space lie in a real linear $(k - 1)$-space, also that this real $(k - 1)$-space exists if and only if the imaginary k-space lies in a real $(k + 1)$-space, and obtains the theorems corresponding to these by duality. A number of theorems follow concerning necessary and sufficient conditions for the determination of real spaces by sets of imaginary elements.

B. A. BERNSTEIN,
Secretary of the Section.

THE APRIL MEETING OF THE SOCIETY IN CHICAGO

The nineteenth western meeting of the Society being the fifty-first regular meeting of the Chicago Section, was held at the University of Chicago on Friday and Saturday, April 13 and 14, 1923. There were about eighty persons present at this meeting, among them the following sixty members of the Society:

Ballantine, E. M. Berry, Bliss, Blumberg, Brahana, Brink, C. C. Camp, Coble, Copenhaver, H. B. Curtis, Curtiss, Denton, Dickson, Dresden, Emch, Feldstein, Glover, L. M. Graves, W. L. Hart, M. G. Haseman, Hildebrandt, Hull, Ingraham, Dunham Jackson, Kazarinoff, Kinney, Krathwohl, Lefschetz, Logsdon, N. B. MacLean, MacMillan, Marshall, T. E. Mason, McEwen, J. V. McKelvey, G. A. Miller, Miser, E. H. Moore, E. J. Moulton, F. R. Moulton, C. I. Palmer, Pounder, Rechard, H. L. Rietz, Roman, Roth, Schottenfels, Sinclair, Slaught, Edwin R. Smith, Stetson, R. B. Stone, J. H. Taylor, E. L. Thompson, Townsend, J. S. Turner, Van Vleck, F. E. Wood, Yanney, J. W. A. Young.

The meeting was opened at 10 A.M. on Friday by Professor E. H. Moore, who presided during the first part of the morning session, after which Professor D. R. Curtiss took the chair. During the sessions of Friday afternoon and Saturday morning Professor A. B. Coble, Chairman of the Section, presided. On Friday afternoon Professor S. Lefschetz gave a symposium lecture on *Curves traced on algebraic surfaces*; this lecture will appear in the June number of this BULLETIN. Upon motion of Professor E. H. Moore, the Section expressed to Professor Lefschetz its appreciation of this lecture. The Section voted to hold the Christmas meeting of 1923 in Cincinnati, in affiliation with the meetings of the American Association for the Advancement of Science.

The papers read at this meeting are listed below. The paper of Mr. Davis was read by Professor Dresden. The papers of Blumberg and Barnett were read by title.

1. Professor H. L. Rietz: *On the representation of a certain fundamental law of probability of Laplace.*

This paper first develops the formula of Laplace for the frequency distribution of the sum of n elements each taken at random from an interval 0 to a by a simple geometrical method involving mathematical induction. It next shows that the approximation which Laplace used in his application of this formula to the problem of the random distribution of the inclination of the orbits of comets is given by the first two non-vanishing terms of the Gram-Charlier representation of an arbitrary continuous frequency function. The paper next develops a more accurate approximation from the Gram-Charlier representation in the sense of a certain least squares criterion. The main difficulty and much of the interest in the paper consists in obtaining remarkably simple expressions for moments of area under our theoretical frequency curve. The auxiliary theorems proved in this connection seem to be of interest in combinatorial analysis.

2. Professor F. E. Wood: *An application of areal coordinates to mixtures and the graphical solution of equations.*

In this paper the author derives the well known results involving the use of points within an equilateral triangle to represent mixtures of three substances; a different method of approach enables him to extend the representation to diluted mixtures of three substances and to general triangles. Some theorems, apparently new, are proved and a graphical method is obtained for finding the percentages of three given mixtures which need to be taken to form a desired mixture of the three fundamental substances. Another application is a method of finding graphically the solution of three simultaneous linear non-homogeneous equations in three unknowns.

3. Dr. H. R. Brahana: *A theorem concerning unit matrices with integral elements.*

Given a matrix M, the normal form of a skew-symmetric matrix of determinant 1 and with integral elements. Poincaré stated without proof that any matrix A with integral elements which satisfies the relation $AMA' = M$ can be considered as a product of matrices of two particular types. In this paper the theorem is proved and it is further shown that any matrix D which satisfies the relation $DND' = N$, where N is not the normal form, is equal to $B^{-1}AB$, B being unique for each N.

4. Dr. H. R. Brahana: *Riemann surfaces and the map problem.*

Heawood solved the map problem for a surface of genus 1, and pointed out the method of procedure for $p > 1$. There is required a "map of verification" for the minimum number of colors for each value of the genus greater than 1. Heffter gave these maps for each value up to 6. This paper translates the problem into one of selecting systems of branch points and cuts on a sphere so that the Riemann surface thus defined constitutes the required map on a surface of given genus. The solutions are given for genus equal to 1, 2, and 3. The paper also contains a definition of a Riemann surface based on analysis situs alone.

5. Mr. H. T. Davis: *An application of fractional differentiation to a class of Volterra integral equations.*

A theory of fractional differentiation is developed which removes a difficulty of Riemann's definition. The theory is then used to obtain solutions of Volterra integral equations of the second kind with infinite discontinuities in the kernel.

6. Professor G. A. Miller: *Same left co-set and right co-set multipliers for any given finite group.*

This paper will appear in full in an early issue of this BULLETIN.

7. Dr. M. M. Feldstein: *The invariants of the linear group modulo* p^k.

The n-ary linear homogeneous group modulo p^k, $G(n, p^k)$, is decomposed into factor-groups in two ways. The first decomposition brings to light the relations among the invariants of $G(n, p^k)$ and those of $G(n, p^{k-1})$. The second decomposition serves to derive the necessary and sufficient conditions which the invariants of $G(n, p^{k-1})$ must satisfy in order to be invariant under $G(n, p^k)$. Special treatment is required for $G(n, 2^k)$. A fundamental set of invariants is set down.

8. Professor S. Lefschetz: *Continuous transformations of manifolds.*

In this note there is given a new method for treating questions of continuous transformations of an n-dimensional manifold W_n. In particular it is shown how the determination of the minimum number of fixed points is reduced to a readily solved problem concerning intersections of cycles of a W_{2n}.

9. Mr. M. H. Ingraham: *Certain limitations of the value of the complete independence of a set of postulates.*

The notion of complete independence was introduced partly in order to do away with certain trick methods of securing ordinary independence. It eliminated the use of vacuous fulfilment of a postulate. Since then methods of questionable value, such as strengthening of postulates and introducing irrelevant elements have arisen for use in securing complete independence. Examples of such methods in recent papers are shown as well as examples against which there can be no criticism. The completely independent set of postulates for positive integers which was previously presented by the writer have such undesirable features and another set is presented from which these features have been eliminated.

10. Professor L. E. Dickson: *Foundations and status of the theory of linear algebras.*

In the usual definition of a linear algebra A of order n over a field F, each element of A is an ordered set $(x_1, \cdots, x_n)$ of n numbers of F. In recent work of Scorza (PALERMO RENDICONTI, vol. 45, p. 7, and *Corpi Numerici e Algebre,* Messina, 1921) the elements u_i of A are undefined entities, combined by addition and multiplication, and by a scalar multiplication producing products of u_i and numbers α_i of F, certain associative and distributive laws being postulated for these operations. Another postulate is that A contains n elements u_i such that every element of A is expressible in one and only one way in the form $\sum\alpha_i u_i$. The present paper replaces this strong postulate by the much weaker one that A has a finite basis. Various theorems are proved for algebras A over any field F which have been proved hitherto only when F is the field of all complex numbers. The results of this paper and those of the following paper will appear in the author's book entitled *Algebras and Their Arithmetics,* to be published in July by the University of Chicago Press.

11. Professor L. E. Dickson: *General theory of hypercomplex integers.*

Books on the arithmetic of quaternions were published by R. Lipschitz in 1886 and by A. Hurwitz in 1919. In a long series of extensive memoirs, Du Pasquier attempted to extend Hurwitz's methods to various algebras other than quaternions, but without success since under his definition most aglebras do not have integers, while if integers do exist, they are usually without interest since the laws of arithmetic fail and cannot be restored by the introduction of ideals however defined. This criticism is explained in full in the author's memoir in the JOURNAL DE MATHÉMATIQUES, 1923, where there is presented a new wholly satisfactory theory of integers in any algebra over the field of complex numbers. The present paper extends this theory to algebras over any field.

12. Dr. Irwin Roman: *The optical transformation of surface differentials.*

The problem of geometrical optics is that of studying the new wave train when we know the incident wave train and the refracting surface. The usual law of refraction is contained in the invariance of the vector product of $r_i h_i$ and H_i where r_i is the refractive index of the ith medium, h_i is the unit normal vector of the wave incident on the ith surface, and H_i is the unit normal vector for the interface between the ith and $(i+1)$th media, at the point of incidence. The author combines a vector and a differential geometry method, obtaining the first and second differential coefficients of the refracting surface from the corresponding values for the interface and the incident wave, along with the distance function for this pair. For the special case of pure propagation, without refraction, the results furnish the usual formulas for parallel surfaces. The formulas simplify when the lines of curvature are taken as parametric curves, and also when the surface is one of revolution. A number of special cases are considered.

13. Professor F. R. Moulton: *Solutions of ordinary differential equations.*

If the right members of the differential equations are analytic in the dependent and independent variables, or if they have the Lipschitz property in the dependent variables and are continuous in the independent variable, then it is shown by a process depending on two parameters that the solution exists over the domain of definition of the differential equations. As a by-product the validity of the method of numerical solution of differential equations is established. It is shown by very direct means that the solutions and their first derivatives with respect to the independent variable are continuous functions of the independent variable and of the initial values of the variables, individually and conjointly. The extensions to the cases where the right members of the differential equations have continuous partial derivatives with respect to the dependent variables are included.

14. Professor Dunham Jackson: *On approximation by functions of given continuity.*

The problem of this paper, in its simplest form, is the approximation of an arbitrary continuous function $f(x)$ by means of functions satisfying a given Lipschitz condition. The discussion is concerned with the existence, uniqueness, and convergence properties of the function of closest approximation according to the criterion of least mth powers. The treatment is susceptible of extension in various directions.

15. Mr. D. Kazarinoff: *Vector treatment of the extrema of double integrals.*

In this paper it is shown that a double integral of the type studied by G. Kobb (Acta Mathematica, vol. 16 (1892)) with the integrand satisfying his conditions (4), (ibid., p. 69), can be reduced to $\int\int_S F(x, y, z, X, Y, Z)d\sigma$ and vice versa. Here X, Y, Z are the direction cosines of a normal to S and F is such that, k being any real constant, $XF_X + YF_Y + ZF_Z = kF$. This theorem is a consequence of one given by J. Radon, Monatshefte, vol. 22 (1911), pp. 53–63.

The author considers then the integral $\int\int_S F(r, n)d\sigma$ where F is a scalar function of r, a vector with components x, y, z, and n, the unit vector with components X, Y, Z, and $\text{grad}_n F\cdot n = kF$, and he derives a differential equation which a surface S must satisfy to furnish an extreme value for the integral. The author also considers some particular cases of this equation and its transformation into more explicit form, yielding a geometric interpretation.

16. Professor F. E. Wood: *Cubics associated with a net of conics.*

There is, in general, a unique cubic $K = 0$ associated with a net of conics, with the property that the first polars of the points of the plane with respect to $K = 0$ form the conics of the net. The equation of this cubic, the *fundamental cubic* of the net, and some of its properties, are derived.

If the point $P(\lambda_1, \lambda_2, \lambda_3)$ is associated with the conic $\lambda_1\varphi_1 + \lambda_2\varphi_2 + \lambda_3\varphi_3 = 0$ of a net of conics, then the points P which correspond to the degenerate conics of the net form a cubic curve in the λ-plane. This is called a Δ *cubic* of the net. In this paper the properties of the Δ cubics are obtained, and certain relations existing between the fundamental cubic, the Δ cubics and the Jacobian cubic of the net. The fundamental cubic of a net will, in general, uniquely determine and will be uniquely determined by a net of conics, which is true for the Jacobian cubic or the Δ cubics in special cases only.

17. Professor J. S. Turner: *An extension of the theory of the double modulus.*

In this paper, certain theorems of the classical double modulus theory are extended to the double modulus $m, P(v)$, where

m is any positive integer, and $P(v)$ is a rational and integral function of v, with integral coefficients, irreducible modulo m. Other theorems are extended to the double modulus p^2, $P(v)$, where p is a prime, and $P(v)$ is irreducible modulo p.

18. Professor Henry Blumberg: *On certain properties of sets of positive measure.*

Let $M_1, M_2, \cdots, M_n$ be n planar sets of positive measure (Lebesgue). It is then shown that there exist in the plane n circles $C_1, C_2, \cdots, C_n$ such that, if $P_1, P_2, \cdots, P_n$ are any n points lying respectively in these circles, there is a congruent set of points $Q_1, Q_2, \cdots, Q_n$ lying respectively in $M_1, M_2, \cdots, M_n$. Again, if M is a planar set of positive measure and $S = \{P_1, P_2, \cdots, P_n\}$, any finite set of points in the plane, there exists in M a set similar to S. Other properties are obtained. The results of the paper hold for n-space.

19. Professor I. A. Barnett: *On a class of invariant subgroups of the conformal and projective groups in function space.*

Kowalewski has defined the conformal group in function space as the totality of all regular infinitesimal transformations which leave invariant the angle between two curves in function space. In this paper all the subgroups of the conformal group are found which leave invariant the manifold in function space $\int_0^1 [f^n(x)/k(x)]\, dx = 1$. The explicit forms of these subgroups are given for the cases $n = 1, 2$. It is shown, furthermore, that for $n > 2$, there are no subgroups having the required property. An analogous discussion is made for the projective group of function space.

20. Professor I. A. Barnett: *The area-preserving group in function space.*

This paper studies all the regular infinitesimal transformations which leave invariant the areas of triangles in function space. These are $\delta f(x) = [\alpha(x) + \int_0^1 \beta(x, y)\, f(y) dy]\delta t$, where $\alpha(x)$ and $\beta(x, y)$ are arbitrary continuous functions of their arguments such that $\beta(x, y) + \beta(y, x) = 0$. These transformations form a group in the sense defined by Kowalewski which is, in fact, the analog of the group of motions in n-space. The preceding transformations also leave invariant all the curvatures of a skew curve in function space.

ARNOLD DRESDEN,
Secretary of the Section.

THE APRIL MEETING OF THE SOCIETY IN NEW YORK

The two hundred twenty-ninth regular meeting of the Society was held at Columbia University, on Saturday, April 28, 1923, extending through the usual morning and afternoon sessions. The attendance included the following fifty-seven members:

Archibald, Beal, Birkhoff, Blichfeldt, B. H. Camp, Cole, Crum, Dantzig, Dodd, Douglas, Fine, Fiske, Fite, M. C. Foster, Philip Franklin, Fry, Gafafer, Gill, Gronwall, Hazlett, Hebbert, E. R. Hedrick, Hille, Huntington, Joffe, Kellogg, Kircher, Kuhn, Lamson, MacDuffee, Mathewson, Meder, Mirick, Molina, Mullins, Northcott, Oglesby, Osgood, Pell, Pfeiffer, Raynor, Reddick, R. G. D. Richardson, Ritt, Rosenbaum, Seely, Siceloff, Silverman, Sosnow, Swartzel, Tracey, Vandiver, H. E. Webb, Weisner, Wetzel, H. S. White, J. W. Young.

At the meeting of the Council, the following twenty persons were elected to membership in the Society:

Professor Thomas Cicero Amick, Elon College;
Professor Harry Cyrus Bradley, Massachusetts Institute of Technology;
Mr. Alexander Joseph Cook, Harvard University;
Professor Palmer Hampton Graham, New York University;
Professor David Arthur Hatch, Lafayette College;
Mr. Harold Hotelling, Princeton University;
Professor Emma Hyde, Kansas State Agricultural College;
Mr. Philip Chapin Jones, Goodyear Tire and Rubber Company;
Mr. Ralph Eugene Kennon, University of Iowa;
Mr. Edgar Lucien Larkin, Lowe Astronomical Observatory;
Mr. Ben Zion Linfield, Harvard University;
Mr. George Averett Lyle, Lehigh University;
Mr. Dalip Singh Saund, University of California;
Mr. David Skolnik, Central High School, Newark;
Mr. Han Yih Tsang, Columbia University;
Mr. John Isaac Vass, Northwestern University;
Professor Evelyn Walker, Hunter College;
Mr. Albert Harry Wheeler, North High School, Worcester;
Professor Hugh Brown Wilcox, University of Minnesota;
Mr. Kitizi Yanagihara, Yamagata Higher School.

The Secretary announced that the following members of the London Mathematical Society had entered the American Mathematical Society since the February meeting, under the reciprocity agreement:

Dr. Bevan Braithwaite Baker, University of Edinburgh;

Professor John Edward Aloysius Steggall, University College, Dundee; Professor William Henry Young, University College, Aberystwyth.

Twenty-seven applications for membership in the Society were received.

The Secretary reported the appointment by President Veblen of the following Committee on Endowment: Professors J. L. Coolidge (chairman), Arnold Dresden, and G. C. Evans, and Messrs. Robert Henderson and G. E. Roosevelt (treasurer). The Council approved a statement of uses to which income from the proposed endowment may be devoted. Briefly these include the subsidizing of the TRANSACTIONS, the Colloquium Lectures, and other research publications, together with treatises on advanced topics. A statement from Professor Coolidge outlining the proposed programme of the Committee was read.

Professor Edward Kasner was elected a member of the Editorial Committee of the TRANSACTIONS, as successor to Professor L. P. Eisenhart, who had declined renomination.

Professor Eisenhart was elected to complete the term of service of Professor C. N. Haskins as representative of the Society in the Division of Physical Sciences of the National Research Council, Professor Haskins having asked to be relieved because of ill health.

The Secretary reported that the Cole Prize Fund (see this BULLETIN, vol. 29, p. 14) had been augmented to more than one thousand dollars. The following committee was appointed to set the first prize problem and to arrange the conditions of award: Professors H. S. White (chairman), Blichfeldt, Dickson, Fiske, and Osgood.

About fifty members took luncheon together between the sessions, and fifteen gathered at the dinner after the meeting.

Professor H. S. White presided at the morning session, relieved in the afternoon by Professor E. V. Huntington. Titles and abstracts of the papers read at this meeting follow below. Mr. Michal was introduced by Professor Evans, Professor Nörlund by Professor Birkhoff, Dr. Rainich by Professor Kasner, and Mr. Levy by Professor Eisenhart.

Professor Haskins' paper was read by Professor Silverman, and the papers of Bliss, Hutchinson, Glenn, Gronwall (first paper), Douglas (second paper), Evans and Bray, Michal, Wiener, Walsh, Garabedian, Safford, Graustein, Nörlund, Eisenhart, Brinkmann, Lipka and Hollcroft, Zeldin, and Veblen and Thomas were read by title.

1. Professor W. L. Crum: *The resemblance between the ordinate of the periodogram and the correlation coefficient.*

This paper points out that the ordinate of the periodogram, as now defined, does not differ materially from the product-sum which occurs in the numerator of the correlation coefficient for the original data with a periodic function of the simplest trigonometric type. This fact indicates that a slight change in the definition of the periodogram may be desirable, and suggests also an ideal means of studying the lag in the correlation of two historical series.

2. Professor G. A. Bliss: *Birational transformations simplifying singularities of algebraic curves.*

There is a well known theorem in the theory of algebraic curves which states that every such curve can be transformed by a birational transformation into one which has no singularities except double points with distinct tangents. The theorem is not a simple one to demonstrate, and many of the proofs which have been given are incomplete or inaccurate. In a preceding paper * the author has commented upon these proofs and has signalized two of them as being especially interesting. One is by Walker, who developed an alteration, suggested by Klein in 1894, of a method originally devised by Bertini for the projective plane. In the second, by Hensel and Landsberg, reasoning proposed by Kronecker in 1881 is extended to apply to curves in the function-theoretic plane. Both of these proofs are lengthy and complicated when all the details are taken into consideration. In the present paper the author has remodeled the method of Kronecker so that it can be applied to both planes, and has attained what he hopes will be regarded as simpler proofs of the two corresponding theorems.

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* *The reduction of singularities of plane curves by birational transformations,* this BULLETIN, vol. 29 (1923), pp. 161–183.

3. Professor O. D. Kellogg: *Curvature and the top.*

If a plane or spherical curve has curvature, K, which has a continuous derivative with respect to the arc, s, and if for $a \leqq s \leqq b$ this derivative is positive, the osculating circle at the point s_2 lies within the osculating circle at s_1 if $a \leqq s_1 < s_2 \leqq b$. The present paper proves this theorem, and gives applications to the theory of the top suggested by Professor Osgood's article in the April number of the TRANSACTIONS OF THIS SOCIETY (vol. 23, No. 3).

4. Professor E. V. Huntington: *Simplified proof of l'Hospital's theorem on indeterminate forms.*

L'Hospital's familiar theorem on indeterminate forms is usually based on the generalized law of the mean, the proof of which often appears to the student as a highly ingenious device. The present author's proof of this theorem depends only on the most elementary properties of integrals, and the motive for each step taken is comparatively obvious. The theorem is stated in an extended form so as to include curves with vertical points of inflection, and new examples are given to show the necessity of the condition that the derivative of the denominator must not vanish in the neighborhood of the point in question.

5. Professor E. V. Huntington: *Tables of Lagrangean coefficients, for interpolating without differences.*

The chief advantages of the Lagrangean formula for interpolation (for equal intervals) are (1) that the required interpolated value is expressed directly in terms of the tabulated values, without the necessity of forming a table of differences, and (2) that the formula, unlike those of Stirling, Bessel, etc., can be written down at once, without effort, and verified by inspection. The chief disadvantages are (1) the fact that the numerical factors required are not small numbers as in the case of the ordinary formulas, and (2) that tables of the coefficients have not hitherto been available. The first of these disadvantages has largely disappeared with the advent of computing machines, and the second is removed by the tables presented by the present author. By the aid of these tables an interpolated value can be computed on the machine in less time than it takes to write out the differences. The tables are arranged for polynomials of the third, fourth, and fifth degrees.

6. Professor J. I. Hutchinson: *A remarkable class of entire functions.*

If $f(z) = a_0 + a_1z + \cdots + a_nz^n + \cdots$ has real and positive coefficients and if $a_n^{-1} = b_1b_2 \cdots b_n$, it is proved that $b_n \geqq 4b_{n-1}$ $(n = 2, 3, \cdots)$ are necessary and sufficient conditions that $f(z)$, as well as any polynomial formed by any number of consecutive terms of $f(z)$, has all its roots real and distinct (except $z = 0$, when it is a multiple root). Various properties and special cases are considered.

7. Professor C. N. Haskins: *On Newton's formulas for the sums of the powers of the roots of an algebraic equation.*

The theorem of this note was brought out by an attempt to simplify and check Bourguet's* computation of the coefficients of the power-series development of the Γ function.

8. Professor O. E. Glenn: *Invariants of the transformation of a differential form by analytic transformations.* Preliminary report.

It is known that restrictions which reduce the number of operations of the group or set of transformations applied to a quantic enlarge the totality of independent concomitants. If conditions are placed upon the arbitrary functions in the transformations for a binary differential form, such as to make these functions satisfy the pair of conditions for being analytic, the differential parameters increase in numbers and new methods are required. The author develops theory for the identification of complete systems in certain domains.

9. Dr. T. H. Gronwall: *Mutual induction of two square coils.*

This paper gives a method for the rapid numerical computation of the mutual induction of two square coils with parallel axes, one coil being rotated through any angle about its axis. Such an arrangement is used in a certain type of radio antenna. The paper will appear in the BELL SYSTEM TECHNICAL JOURNAL.

10. Dr. Einar Hille: *On Dirichlet's series with complex exponents.*

The author considers series of the type $\sum a_ne^{-\lambda_n s}$, where the exponents λ_n form a sequence of complex numbers such that $|\lambda_n| \to \infty$. It is shown that the region of absolute convergence of such a series is a convex domain. This region is determined in terms of the a_n and the λ_n under different assumptions concerning the exponents. For every convex region Dirichlet's series can be constructed, having the region as region of absolute convergence. The results are extended to Dirichlet's integrals.

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* ANNALES DE L'ECOLE NORMALE, (2), vol. 10 (1881), pp. 175–233.

11. Dr. Jesse Douglas: *Normal congruences and quadruply infinite families of curves in space.* Second paper.

In a previous paper of the same title (this BULLETIN, vol. 28, p. 238) the author classified quadruply infinite families of space curves with respect to the normal congruences contained within them. In that classification, two types of curve family, termed natural and quasi-natural, were predominant. The present paper solves the following problem: To determine, for each of the following four properties, all families of ∞^4 curves in space which have that property. (1) The ∞^2 curves of the family which pass through an arbitrary point form a normal congruence. (2) The ∞^2 curves of the family which meet an arbitrary plane perpendicularly form a normal congruence. (3) The ∞^2 curves of the family which meet an arbitrary sphere perpendicularly form a normal congruence. (4) The ∞^2 curves of the family which meet an arbitrary straight line perpendicularly form a normal congruence. There are three types with the property (1), of which two are the natural and quasi-natural. Five types, including the natural and quasi-natural, have the property (2). All and only natural and quasi-natural families have the property (3). All and only natural families have the property (4).

12. Dr. Jesse Douglas: *Determination of all families of ∞^4 curves in space in which the sum of the angles of every triangle is two right angles.*

This paper will appear in full in an early issue of this BULLETIN.

13. Dr. C. C. MacDuffee: *On the complete independence of the functional equations of involution.*

The functional equations (1) $\varphi(a, m)\cdot\varphi(a, n) = \varphi(a, m + n)$, (2) $\varphi(\varphi(a, m), n) = \varphi(a, mn)$, (3) $\varphi(a, n)\cdot\varphi(b, n) = \varphi(ab, n)$ are satisfied unreservedly by the numbers of the class C_+ of positive numbers and zero, when $\varphi(a, n)$ is identified with the power function a^n. The complex numbers satisfy these equations when proper determinations of the multiple-valued power functions are used. In this paper, a non-linear algebra

N is introduced whose elements are the pairs of real numbers $[\xi, r]$. Two operations, addition and multiplication, are defined, and it is shown that these operations obey all the ordinary laws of algebra except the associative law of addition. This algebra N is then used in the discussion of the complete independence of equations (1), (2) and (3). With certain underlying conditions, it is proved that the functional equations are not completely independent, for (1) and (2) imply (3). The remaining seven cases are non-empty, for seven functions are defined on the algebra N which obey the underlying conditions, each fulfilling one of these seven cases.

14. Professor G. C. Evans and Dr. H. E. Bray: *The Dirichlet problem for the sphere, and its generalization; necessary and sufficient conditions.*

The authors find necessary and sufficient conditions that a function harmonic within a sphere, not necessarily bounded, may be written as a Poisson integral in terms of boundary values f summable on the surface of the sphere in the Lebesgue sense; also, more generally, as a Poisson integral in terms of the differential of an additive function of point sets $F(\omega)$ on the surface of the sphere. Extending the result of Fatou, they show that the harmonic function takes on the value of the point set derivative of $F(\omega)$ wherever the latter exists, and hence, in the former problem, the value of f almost everywhere. A physical interpretation in terms of sources and doublets is given for the extended problem, using a new theorem of Vitali on the analysis of continuous functions of limited variation with zero derivative almost everywhere.

15. Mr. A. Michal: *Integro-differential invariants of one-parameter groups of Volterra transformations.* Preliminary report.

The author commences the study of a problem allied to the calculus of variations, by showing that there always exist functionals of a function y and its derivative which are invariant under a given arbitrary continuous one-parameter group of linear functional transformations of Volterra type of the argument y. The functional is assumed to involve y and its derivative between the values 0 and x, and, considered as a functional of these two arguments, may for a given functional value of the derivative of y be assigned arbitrarily as a functional of the other argument.

16. Dr. Norbert Wiener: *A generalization of the Dirichlet problem.*

The author uses the methods of generalized integration due to Daniell to extend the notion of a harmonic function corresponding to given boundary conditions to discontinuous boundary conditions of a very general type. Existence theorems are given.

17. Dr. Norbert Wiener: *A new type of summability.*

The author develops a type of summability with trigonometric weighting factors, and shows that it correctly evaluates the Fourier series of a continuous function. This type of summability is not stronger than the first type of summability of Cesàro.

18. Dr. J. L. Walsh: *On the location of the roots of Lamé's polynomials.*

A certain problem in the statical equilibrium of particles arises naturally in connection with the theory of equations. This same problem, in a generalized form, arises also in connection with the location of the roots of certain polynomial solutions of a differential equation considered by Lamé. This paper extends to the latter type of problem some simple results recently established for the former type.

19. Mr. C. A. Garabedian: *Rods of constant or variable circular cross section.*

In a paper on *Circular plates of constant or variable thickness,* presented to the Society February 24, 1923, the author developed a method of series in elasticity, and called attention to the possibility of treating by the same method the problem of the thin rod. Cylindrical coordinates are adopted, and the rod is generated by revolution of a curve $r = f(z)$ about the axis of z. The chief interest lies in the fact that the section may vary with z. Writing $r = \rho\tau$, $f(z) = \varphi(z)\tau$, it is assumed that the displacements are developable in ascending powers of the parameter τ. The equations of equilibrium become identities on τ, and computation of the coefficients in the formulas of displacement is direct. In prescribing boundary conditions at the ends, conventional use is made of Saint-Venant's principle. The total differential equations which arise can be integrated explicitly in a wide range of cases. Applications are made to the uniformly tapering rod, to a bulging rod, and to the right circular cylinder.

20. Professor F. H. Safford: *A pendulum of varying length.*

This paper was written in answer to a request to obtain the law of variation in length if the amplitude of swing is to increase as rapidly as possible. In solving the differential equation of motion, it was found that a change of the independent variable time could be made which reduced the problem to that of a simple pendulum of fixed length. The amplitude may be made any assigned function of the time, and a criterion was obtained concerning the transition from vibration to complete revolution.

21. Professor W. C. Graustein: *Note on a certain type of ruled surface.*

This paper will appear in full in an early issue of this BULLETIN.

22. Professor N. E. Nörlund: *On certain difference equations.*

The object of this paper is to study the solutions of the difference equation $\Delta^n \omega F(x) = \varphi(x)$, where $\varphi(x)$ is a given function. By a certain method of summation the author defines an operation which is inverse to the difference operation Δ. He finds thus a particular solution of the above equation. This solution may also be defined by boundary value conditions. The paper contains a detailed discussion of the properties of this particular solution. The general solution differs from the above particular solution by an arbitrary function whose nth difference is zero.

23. Professor L. P. Eisenhart: *Orthogonal systems of hypersurfaces in a Riemann space.*

Let $g_{rs}dx^r dx^s$ be the fundamental quadratic form of a Riemann space of order n, and α_{rs} the covariant components of any symmetric tensor of the second order other than g_{rs}. The equations $(\alpha_{rs} + \rho g_{rs})\lambda^r = 0$ determine an n-uple of congruences whose directions at any point are mutually orthogonal, the functions $\lambda_h{}^r$ being the contravariant components of the tangents to a curve of the congruence C_h at any point. The necessary and sufficient conditions upon the tensors g_{rs} and α_{rs} in order that the congruences be normal, and consequently the space admit an n-uply orthogonal system of hypersurfaces, are determined in this paper. It is shown that the problem is an algebraic one. The conditions are simple when the roots of the equation $(\alpha_{rs} + \rho g_{rs}) = 0$ are simple or double at most, but are quite involved where there are roots of the third and higher orders.

24. Professor L. P. Eisenhart: *Symmetric tensors of the second order whose first covariant derivatives are zero.*

The following theorem is established: A necessary and sufficient condition that a Riemann space admit a symmetric covariant tensor of the second order α_{rs} other than the fundamental tensor of the space, g_{rs}, such that its first covariant derivative is zero, is that the fundamental quadratic form $g_{rs}dx^r dx^s$ be reducible to a sum of quadratic forms φ^i, the coefficients of each form being functions at most of the x's of that form; then $\alpha_{rs}dx^r dx^s = \sum_i \rho_i \varphi^i$, where the ρ's are arbitrary constants. The determination of whether or not a given space is of this kind is reducible to a problem of algebra.

25. Dr. G. Y. Rainich: *Geometry of curved space without coordinates.*

An n-dimensional bundle of vectors is introduced axiomatically, the properties of addition of vectors, multiplication of vectors with scalars, and formation of ratio serving as axioms. Space is considered as a totality of points; an axiom states that vectors issuing from one point constitute an n-dimensional bundle. This permits introducing a relation between different bundles. The transitivity of this relation is the condition that space be euclidean. In the general case the geometry in the vicinity of every point can be given by a trilinear vector function which is the generalized Riemann tensor. Scalar and vector multilinear functions are introduced as two kinds of tensors which are essentially distinct unless a length unit is introduced. Weyl's "Dichten" are shown on the contrary not to be essentially different from tensors. The notions of transformations, invariants, covariant and contravariant quantities are not essential to geometry and arise only with the introduction of coordinates. The metric does not define the relation between bundles, but for the representation of a metric we might use these relations, and in this case the Riemann tensor can be deduced from this representation.

26. Mr. Harry Levy: *Normal congruences of curves in a Riemann space.*

The following theorems are demonstrated: (1) A necessary and sufficient condition that a congruence of curves in a Riemann space of order n admit a family of r-dimensional ($r > 1$) hypersurfaces as orthogonal trajectories is that

$\gamma_{ijk} = \gamma_{ikj}$ $(i = 1, 2, \cdots, n - r;\ j, k = n - r + 1, \cdots, n)$, where the functions γ_{ijk} have the significance given them by Ricci and Levi-Civita (MATHEMATISCHE ANNALEN, vol. 54 (1901), p. 148). (2) If each of n mutually orthogonal congruences has a family of r-dimensional hypersurfaces as orthogonal trajectories, and if the n families of hypersurfaces are distinct, the congruences are normal, that is, each has a family of $(n - 1)$-dimensional hypersurfaces as orthogonal trajectories.

27. Dr. Philip Franklin: *Linear tensor equations.*

In this paper we investigate under what conditions a set of equations involving tensor components, invariant, as a set, under changes of coordinates, is equivalent to a set of tensor equations, i.e., equations expressing the vanishing of *all* the components of a tensor. We show that a set of equations *linear* in the components of a single tensor, with numerical coefficients, holding for all coordinates, is equivalent to a set of tensor equations. This theorem, proved for n-space, is applied to the proof that there are essentially only three distinct linear tensor equations in the curvature tensor, a theorem proved for 4-space by Birkhoff. A general theorem restricting the possible forms of linear tensor equations is added, which may be used to deduce such equations as those of the electromagnetic field.

28. Dr. Philip Franklin: *A qualitative definition of the potential functions.*

The purpose of this paper is to set up assumptions characterizing the potential functions which do not involve derivatives or integrals, and express obvious properties of the physical quantities giving rise to potential functions. It is shown that a class of functions of two variables such that its members are continuous, any linear combination of two members is in the class, any member gives rise to a new member if the coordinate axes are shifted (orthogonal transformation of the variables), it contains a constant function, and for an infinite sequence of functions of the class, taking values on the boundary of a fixed circle approaching zero uniformly over the boundary, the values at the center of the circle cannot approach a limit different from zero, *is necessarily a set of harmonic functions.* An analogous set of assumptions is given for potential functions of three variables.

29. Mr. H. W. Brinkmann: *Riemann spaces conformal to Einstein spaces.*

A Riemann n-space which satisfies those Einstein equations often referred to as the "cosmological equations" is called an *Einstein* space. The author obtains a necessary and sufficient condition that a given Riemann space be conformal to such a manifold. This is then applied to the special case where the given manifold is an Einstein space to start with. For the dimensionality $n = 4$ there is proved, among other things, the following theorem: If two four-dimensional Einstein spaces are mapped conformally they are either both of constant Riemann curvature (spherical spaces) or they are isometric and the conformal map in question is merely a change of scale. A note on this subject has been sent to the PROCEEDINGS OF THE NATIONAL ACADEMY OF SCIENCES.

30. Mr. H. W. Brinkmann: *Einstein spaces mapped conformally on each other.*

In this paper the author determines all n-dimensional Einstein spaces which can be mapped conformally on another Einstein space provided that map which consists in a mere change of scale is ignored. The results are quite simple, and agree for $n = 4$ with the theorem stated in the preceding paper. It must be noted that the two Einstein manifolds in question need not have the same scalar curvature, but if their scalar curvatures are both zero or both not zero it is shown that one of the two conformal spaces is isometric to that obtained from the other by a suitable change of scale. The isometric map is, however, not established by the conformal one. When one of the scalar curvatures vanishes and the other does not, it is evident that the two spaces can not be isometric even if a change of scale is allowed.

31. Dr. M. C. Foster: *Surfaces with orthogonal loci of the centers of geodesic curvature of an orthogonal system.*

The displacements of the centers of geodesic curvature are considered relative to the moving trihedral. The loci of these points are not surfaces but orthogonal curves of which at least one must be a straight line. The given orthogonal system must be the lines of curvature. Such surfaces are *surfaces of Joachimsthal* with circular lines of curvature in one system, and for which the fundamental quantities satisfy the relation $(\partial/\partial u)((\partial r/\partial u)/\xi) + \xi r_1 = 0$.

32. Professor Joseph Lipka: *Trajectory surfaces.*

In a space of any dimensionality V_n, any two directions through a point P determine a pencil of directions, and the curves through P in the pencil of directions which minimize the integral $\int \phi ds$, where ϕ is any point function and ds is the element of length, determine a surface or spread of two dimensions, called a trajectory surface (a generalization of a geodesic surface). In this paper the author discusses the Gaussian curvatures of these surfaces and their relations to the Gaussian curvatures of the corresponding geodesic surfaces. It is found, e.g., that a necessary and sufficient condition that the curvatures of a trajectory surface and its corresponding geodesic surface should be the same is that the determining pencil of directions contain the direction of the vector ϕ. Furthermore, we have here a generalization of Ricci's median curvature, principal directions, principal congruences, and principal invariants in any space V_n; the theorems are analogous to those which Ricci has found in a discussion of geodesic surfaces.

33. Professor Joseph Lipka: *Geometric interpretation of the second differential parameter.*

Given any surface and any function ϕ of the coordinates on the surface, the invariant or second differential parameter of ϕ, usually designated by $\Delta_2\phi$, plays an important rôle in many problems in surface theory. The present paper contains a simple geometric interpretation of this parameter, which may be stated as follows: If any direction on the surface through a point P moves first by parallelism and then by conformal parallelism (whose characteristic function is ϕ) completely around an infinitesimal cycle drawn on the surface through P, the ratio of the angle formed by its final positions to the area of the cycle is the value at P of $\Delta_2\phi$.

34. Professor E. L. Dodd: *Formulas for the greatest and the least variate under general laws of frequency or error.*

The author classifies probability or relative frequency functions $\phi(x)$ as follows: For values of x sufficiently large, $\phi(x)$ is expressed as the product of an arbitrary $\psi(x)$, which satisfies certain inequalities, and (1) 0; (2) $x^{-1-\alpha}$; (3) g^{x^α}; (4) $g^{(\log_c x)^\gamma}$; (5) g^{c^x}; (6) x^{-x}; with $\alpha > 0$, $0 < g < 1$, $c > 1$, $\gamma > 1$; and then shows that for any positive η, there is for large enough n, a probability $> 1 - \eta$ that the greatest of n variates will be: (1) Constant, (2) $n^{1/\alpha}$; (3) $(-\log_g n)^{1/\alpha}$,

(4) c to the power $(-\log_g n)^{1/\gamma}$, (5) $\log_c (-\log_g n)$, (6) G in $G^G = n$, respectively; with a relative error small at pleasure for the values in (1), (3), (5), (6); and for $1/\alpha$ and $1/\gamma$ in (2) and (4). These results are applied to the seven Pearson types, the Bruns series (finite), the Jorgensen logarithmic function, the Poisson exponential (law of small numbers), the Charlier B-series (finite), and the Makeham life function. If G is determined from $\int_G^\infty \phi(t)dt = 1 - 2^{-1/n}$, it is, of course, equally probable that the greatest variate will or will not exceed G. This gives results for $ke^{-h^2x^2}$ in close agreement with those of Bortkiewicz.*

35. Professor T. R. Hollcroft: *Singularities that may be added to those of curves of given order.*

This paper will appear in an early issue of this BULLETIN.

36. Dr. S. D. Zeldin: *On the quadratic ternary partial differential equations admitting Lie groups of orders* 4 *and* 5.

The author determines the coefficients of the linear partial differential equation of the second order in three independent variables of the form $a_{11}z_{11} + a_{22}z_{22} + a_{33}z_{33} + 2a_{12}z_{12} + 2a_{13}z_{13} + 2a_{23}z_{23} + 2a_1z_1 + 2a_2z_2 + 2a_3z_3 + az = 0$ which is invariant under Lie groups of orders 4 and 5 of given structures.

37. Professor H. F. Blichfeldt: *On the approximate solution in integers of a set of m linear non-homogeneous equations in $n > m$ unknowns, and the final form of Kronecker's theorem.*

Kronecker's theorem asserts that if the coefficients of the equations in question satisfy certain "conditions of rationality," then a set of integers $x_1, x_2, \cdots, x_n$ exist which, when substituted for the unknowns, will satisfy the equations up to certain errors whose absolute values are all less than a previously assigned small positive quantity ϵ. In the present paper it is shown that if the errors are designated $\epsilon_1, \cdots, \epsilon_m$, and if $R = \sqrt{x_1^2 + \cdots + x_n^2}$, we may demand, in addition to $|\epsilon_i| < \epsilon$, also that $|\epsilon_1\epsilon_2\cdots\epsilon_m|R < f$, a certain number depending upon the coefficients of the given equations, but independent of ϵ and of $x_1, \cdots, x_n$. When $m = 1$, f represents any preassigned positive number. It is furthermore shown that when $m > 1$, equations exist for which nothing may be demanded of the errors individually, beyond $|\epsilon_i| < \epsilon$; the case $m = 2$, $n > 3$ may furnish a possible exception.

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* *Variationsbreite und mittlerer Fehler*, SITZUNGSBERICHTE DER BERLINER MATHEMATISCHE GESELLSCHAFT, vol. 21; October, 1921.

38. Professor Oswald Veblen and Mr. T. Y. Thomas *Geometries of paths admitting first integrals.*

In this paper, the condition for the existence of homogeneous linear and quadratic integrals of paths in the geometry of paths introduced by Professors Eisenhart and Veblen (PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 8 (1922), pp. 19–23) is reduced to the algebraic consistency of a set of equations, as is also the condition of existence of the general first integral subject to the particular restriction that its first covariant derivative vanish. Thus a necessary and sufficient condition for the geometry of paths to reduce to the Riemann geometry is given by the algebraic consistency of a set of equations. Other applications are also made. In addition the paper contains a discussion of normal coordinates for the geometry of paths from a different viewpoint from that adopted by Professor Veblen (PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 8 (1922), pp. 192–197) and a treatment of the subject of covariant differentiation in the geometry of paths in which a whole group of tensors derivable from a given tensor is obtained. The first tensor of this group is the ordinary covariant derivative. The paper will appear in the TRANSACTIONS OF THIS SOCIETY.

39. Dr. T. H. Gronwall: *Isothermal surfaces with spherical lines of curvature in one system.*

By the use of pentaspherical coordinates, it is shown that all these surfaces are obtainable by inversion from the isothermal surfaces with plane lines of curvature in one system, which were investigated by Darboux (*Théorie des Surfaces*, vol. 4, chapter X). The present method determines all such surfaces; it yields Darboux's results in a simpler manner, and also a new class of surfaces of this kind with simple geometrical properties.

40. Dr. T. H. Gronwall: *Extension of Tchebychef's statistical theorem.*

In the first part of this paper, inequalities stronger than that of Tchebychef are found under the assumption that all moments of even order $m_2, m_4, \cdots, m_{2k}$ are known, k being any integer. No assumption on the probability function is made (except that it is never negative). In the second part, a similar investigation is carried out assuming the probability function monotone decreasing beyond the limit of skewness.

R. G. D. RICHARDSON,
Secretary.

AN ELEMENTARY PROOF OF A FUNDAMENTAL LEMMA CONCERNING THE LIMIT OF A SUM*

BY H. J. ETTLINGER

It is the object of this note to give a simple proof of the following lemma upon which may be based the theory of the Riemann integral.

Let the interval $I : a \leqq x \leqq b$ *be divided into* n *equal sub-intervals,* $I(i, n)$, *each of length* $\Delta_n x = (b - a)/n$. *To each* $I(i, n)$ *there corresponds a number* $h(i, n)$. *Let* $|h(i, n)| \leqq M$ *for all values of* i $(\leqq n)$ *and* n, *where* M *is a constant. If* x *is any fixed value in* I, *then for each value of* n, x *is contained in at least one of the sub-intervals. Let it be designated by* $I(x, n)$. *For each fixed* x *and each subdivision* $I(x, n)$ *let the corresponding number be* $h(x, n)$, *and let* $\lim_{n \to \infty} h(x, n) = 0$. *Then*

$$\lim_{n \to \infty} \sum_1^n h(i, n)\Delta_n x = 0.$$

In a recent paper † the above lemma was stated in terms of area as a geometric theorem and used to develop the theorems on definite integrals of fundamental importance in the integral calculus. In addition, applications were given to the transformation of a double integral by change of variables and to the Fredholm method of solution of an integral equation.

A proof based on a generalization of a theorem due to W. H. Young ‡ has been given by R. L. Moore.§ For the proof of the above lemma, however, it is sufficient to use Arzelà's ‖ "lemma fondamentale" which may be stated in terms of the

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[*] Presented to the Society, September 7, 1922.

[†] *A simple form of Duhamel's theorem and some new applications,* American Mathematical Monthly, vol. 29 (1922), p. 239.

[‡] *Open sets and the theory of content,* Proceedings of the London Society, (2), vol. 2 (1904), pp. 20–22.

[§] *On Duhamel's theorem,* Annals of Mathematics, (2), vol. 13 (1912), p. 163.

[‖] *Un teorema intorno alle serie di funzioni,* Lincei Rendiconti, (4), vol. 1 (1885), pp. 262–267.

notation of the above theorem as follows:

If for each n, $I(n) = \sum_1^k I(m_i, n)$, where m_i is a positive integer such that

$$1 \leqq m_1 < m_2 < \cdots < m_k \leqq n,$$

and $I \geqq I(n) > \delta > 0$ for every n, there is at least one point P which is common to an infinite number of the $I(n)$'s.

The proof of the lemma may also be based on the following theorem of Osgood.*

If $s_n(x)$ *is continuous in* $a \leqq x \leqq b$ *and converges when* $n \to \infty$ *for every* x *in the interval, and if* $|s_n(x)| \leqq M$ *for all values of* $n = 1, 2, \cdots$ *and* x *in* $a \leqq x \leqq b$, *where* M *is a constant, then*

$$\lim_{n\to\infty} \int_a^b s_n(x)dx = \int_a^b \lim_{n\to\infty} s_n(x)dx.$$

This theorem is applied as follows. On $I(i, n)$ as base erect an isosceles triangle whose altitude is $2h(i, n)$. The equal legs of these triangles form the graph of $s_n(x)$. Identify

$$\sum_1^h h(i, n)\Delta_n x = \int_a^b s_n(x)dx.$$

Finally note that $\lim_{n\to\infty} s_n(x) = 0$ for every fixed value of x, and the lemma is proved.

Other proofs of Osgood's theorem have been given by F. Riesz,† Bieberbach,‡ and Landau.§ The method given by Landau is substantially that given by Moore (loc. cit.) and makes use of Arzelà's "lemma fondamentale" (loc. cit.). The proof by Bieberbach is essentially the same as that of Landau, with the exception that a simplified proof of Arzelà's theorem is included. The method of Riesz is more elementary and has suggested the proof given in this note.

It is to be noticed *mutatis mutandis* that Osgood's theorem may be regarded as a corollary of the lemma of this note. It would seem to be more natural to establish the lemma stated above by elementary methods and derive from it the properties of the Riemann integral, of which the theorem of Osgood would be one.

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* *On non-uniform convergence and integration of a series term by term*, AMERICAN JOURNAL, vol. 19 (1897), p. 188.

† *Ueber Integration unendlicher Folgen*, JAHRESBERICHT DER VEREINIGUNG, vol. 26 (1917–18), pp. 274–278.

‡ *Ueber einen Osgoodschen Satz aus der Integralrechnung*, MATHEMATISCHE ZEITSCHRIFT, vol. 2 (1918), pp. 156–157.

§ *Ein Satz ueber Riemannsche Integrale*, MATHEMATISCHE ZEITSCHRIFT, vol. 2 (1918), pp. 350–351.

We proceed to the proof. Let

$$f(n) = \sum_1^n |h(i, n)| \Delta_n x.$$

Then

$$f(n) \geqq \left| \sum_1^n h(i, n) \Delta_n x \right| \geqq 0.$$

If $\lim_{n\to\infty} f(n) = 0$ is proved, the lemma is established.

CASE 1. Let $|h(x, n+1)| \leqq |h(x, n)|$ for every fixed x in I. Then $f(n) \geqq f(n+1) \geqq 0$, or $f(n)$ is a monotonically non-increasing function of n. Hence

$$\lim_{n\to\infty} f(n) = A \geqq 0.$$

Divide I into two equal parts. Then $|h(i_2, 2)| \Delta_2 x \geqq A/2$, where i_2 has at least one of the two values, $i = 1, 2$. Divide $I(i_2, 2)$ into two equal parts. Then $|h(i_4, 4)| \Delta_4 x \geqq A/2^2$, where i_4 has at least one value of the set 1, 2, 3, 4. If this process be continued k times, we have $|h(i_m, m)| \Delta_m x \geqq A/m$, where $m = 2^k$ and i_m has at least one value of the set 1, 2, $\cdots$, 2^k. But $\Delta_m x = (b-a)/m$. Hence $A \leqq |h(i_m, m)|(b-a)$. The closed sub-intervals $I(i_2, 2)$, $I(i_4, 4)$, $\cdots$, $I(i_m, m)$ are so related that each is contained in the preceding, and, as n increases without limit, $\Delta_m x$ approaches zero. Therefore there exists a point $\overline{P}$ of I whose abscissa is $\bar{x}$ such that $\overline{P}$ is common to all these sub-intervals. Hence $h(i_2, 2)$, $h(i_4, 4)$, $\cdots$ form a set of numbers $h(\bar{x}, n)$ corresponding to $\bar{x}$. But $\lim_{n\to\infty} |h(\bar{x}, n)| = 0$. Hence $A = 0$. This same method with a slight modification can be used for this case if the length of the subdivisions are unequal.

CASE 2. For the general case, let $h(x, n, m)$ for $n \leqq m$ be the greatest of the numbers $|h(x, n)|$, $|h(x, n+1)|$, $\cdots$, $|h(x, m)|$. Form

$$\phi(n, m) = \sum_1^n h(i, n, m) \Delta_n x,$$

where $h(i, n, m)\Delta_n x$ is the area of the rectangles whose bases fill up $I(i, n)$ and whose heights $h(i, n, m)$ are chosen so as to satisfy the preceding condition for each x in $I(i, n)$. Now $h(x, n, m) \leqq M$ for all values of m and $n \leqq m$. Hence

$$\phi(n, m) \leqq M(b - a).$$

For fixed x and n, $h(x, n, m) \leqq h(x, n, m + 1)$. Hence

$$\phi(n, m) \leqq \phi(n, m + 1).$$

So that for a fixed n, $\phi(n, m)$ is a monotonically non-decreasing function of m. Hence $\lim_{m\to\infty} \phi(n, m) = H(n) \leqq M(b - a)$, and

$$\phi(n, m) \leqq H(n). \tag{1}$$

But $f(n) \leqq \phi(n, m)$, or $f(n) \leqq H(n)$. Hence $\lim_{n\to\infty} H(n) = 0$ will carry with it $\lim_{n\to\infty} f(n) = 0$.

Given $\delta > 0$ and arbitrarily small, define a sequence of positive integers, m_i, to satisfy the inequality

$$\phi(i, m_i) \geqq H(i) - \frac{\delta}{2^i}, \qquad (i = 1, 2, \cdots, n). \tag{2}$$

Since $\phi(i, m) \geqq H(i) - (\delta/2^i)$ for $m > m_i$ we may choose the set m_i to satisfy $m_1 < m_2 < m_3 < \cdots < m_n < \cdots$. Let $\bar{h}(x, n)$ denote the smallest of the numbers $h(x, 1, m_1)$, $h(x, 2, m_2), \cdots, h(x, n, m_n)$. Form

$$\psi(n) = \sum_1^n \bar{h}(i, n)\Delta_n x,$$

where, as above, $\bar{h}(i, n)\Delta_n x$ is the area of the rectangles whose bases fill up $I(i, n)$ and whose heights $\bar{h}(i, n)$ are chosen so as to satisfy the preceding condition for each x in $I(i, n)$.

We shall prove by induction that

$$\psi(n) > H(n) - \delta\left(1 - \frac{1}{2^n}\right). \tag{3}$$

For $n = 1$ we have $\bar{h}(x, 1) \equiv h(x, 1, m)$, and hence $\phi(1, m_1) \equiv \psi(1)$. Hence by (2), (3) is true for $n = 1$. If (3) is not true for every positive integer, let k be the first integer for which it is not true. Then (3) is true for $n = k - 1$, or

$$\psi(k - 1) > H(k - 1) - \delta\left(1 - \frac{1}{2^{k-1}}\right). \tag{4}$$

Now

$$\bar{h}(x, k) \leqq \bar{h}(x, k - 1), \quad \text{and} \quad \bar{h}(x, k) \leqq h(x, k, m_k). \tag{5}$$

But

(6) $\bar{h}(x, k-1) \leqq h(x, k-1, m_{k-1}) \leqq h(x, k-1, m_k)$.

Compare

$$\bar{h}(x,\ k-1) - \bar{h}(x,\ k) \geqq 0$$

and

$$h(x,\ k-1,\ m_k) - h(x, k, m_k) \geqq 0.$$

If $\bar{h}(x, k-1) \leqq h(x, k, m_k)$, then $\bar{h}(x, k-1) - \bar{h}(x,\ k) = 0$, and hence in this case

$$\bar{h}(x, k-1) - \bar{h}(x, k) \leqq h(x, k-1, m_k) - h(x, k, m_k).$$

If $\bar{h}(x, k-1) > h(x, k, m_k)$, then

$$\bar{h}(x, k) = h(x, k, m_k). \tag{7}$$

Subtracting (7) from (6), we have

$$\bar{h}(x, k-1) - \bar{h}(x, k) \leqq h(x, k-1, m_k) - h(x, k, m_k).$$

Hence in every case

(8) $\bar{h}(x, k) \geqq \bar{h}(x, k-1) + h(x, k, m_k) - h(x, k-1, m_k)$,

for every x in I. Then

$$\psi(k) \geqq \psi(k-1) + \phi(k, m_k) - \phi(k-1, m_k). \tag{9}$$

By (4), (2), and (1) we may strengthen (9):

$$\psi(k) \geqq H(k-1) - \delta\left(1 - \frac{1}{2^{k-1}}\right) + H(k) - \frac{\delta}{2k} - H(k-1),$$

or

$$\psi(k) \geqq H(k) - \delta\left(1 - \frac{1}{2k}\right)\cdot$$

This proves (3). Now strengthen (3) and write

$$\psi(n) > H(n) - \delta. \tag{10}$$

But

$$\bar{h}(x, n) \leqq h(x, n, m_n). \tag{11}$$

Also

$$h(x, n, m) = |h(x, \nu(x))| \tag{12}$$

where $n \leqq \nu(x) \leqq m_n$ and $\lim_{n \to \infty} \nu(x) = \infty$.

Now $\bar{h}(x, n)$ for every fixed x in I is a monotonically non-increasing function by (5), and $\lim_{n\to\infty} \bar{h}(x, n) = 0$ by (11) and (12). Hence we have, by Case 1, $\lim_{n\to\infty} \psi(n) = 0$ and $\lim_{n\to\infty} H(n) = 0$, by (10).

University of Texas

KLEIN'S COLLECTED PAPERS, VOLUME II.

Felix Klein: Gesammelte mathematische Abhandlungen. Vol. II: *Anschauliche Geometrie; Substitutionsgruppen und Gleichungstheorie; Zur mathematischen Physik.* Edited by R. Fricke and H. Vermeil. Berlin, Julius Springer, 1922. vi + 713 pages; 185 figures.

Although one of the editors has been replaced in the second volume, the purpose has not been altered, and it appears under the same plan as the preceding one. The memoirs in the first part, mentioned under the first subtitle, are sixteen in number, being memoirs XXXIV to XLIX inclusive. Of these, nine were written during the years 1873–6, five in the interval 1892–5, and two since 1900. This first part occupies 250 pages.

The first memoir is a description of four models of the Plücker complex surface, constructed while the author was a student and assistant of Plücker in Bonn. The first is the general Kummer surface with sixteen nodes, which appears as the surface of singularities of a general quadratic line complex. The second is the surface enveloped by lines of the complex which meet a given line. The line is double on the surface, which has eight double points. The third is defined in the same way, when the given line is an ordinary line of the complex, and the fourth has for double line a singular line of the complex. These models were rather crudely constructed and very crudely reproduced in metal casts, but their importance in the study of singularities of algebraic surfaces is very great. About two pages have been added to the original text, which was prepared for the *Katalog mathematischer Modelle . . .*, von W. Dyck (1892).

The next paper, of 52 pages, is on cubic surfaces. To the original paper that appeared in volume 6 of the MATHEMATISCHE ANNALEN, eighteen pages have been added, which explain more in detail the surfaces with biplanar points, and discuss the configuration of the 27 real lines on the diagonal surface. These two papers are the only ones on the construction of models, but they are sufficient to show the intimate relation between spatial intuition and mathematical conclusion as they existed in the mind of the author.

The procedure is nearly always the same, starting from a singular or indeed composite surface and seeing what changes necessarily follow when one or more parameters change slightly. The method was sharply attacked when first employed, then reluctantly tolerated on account of its unquestioned results, until now it is accepted as a completely rigorous and trustworthy process, brought to a wonderfully high degree of perfection by Italian geometers. The Rodenberg collection of models of cubic surfaces, proposed by Clebsch, was made more complete and systematic by the use of the above memoirs.

When Klein was called to Erlangen, in his opening address he proposed the establishment of a general collection of models and apparatus to simplify the teaching of mathematics. Later the same idea was developed much further in Munich, in collaboration with Professor A. Brill, whose brother L. Brill undertook the commercial manufacture of the models. While in Munich, Klein had for his assistant W. Dyck, who was an important factor in having models and apparatus recognized as an important and necessary part of mathematical instruction. The Brill factory in Darmstadt, later taken over by M. Schilling in Halle, under the scientific direction of Professor F. Schilling has supplied a large proportion of all the plaster models in use throughout the world. Since the death of Martin Schilling, and particularly since the war, the manufacture of a considerable number of the more complicated models has been discontinued. If this state of affairs is not corrected, it will mean a terrible loss to later generations of mathematical teachers.

The Dyck catalogue and a considerable collection of models was sent to Chicago as a part of the German exhibit at the World's Fair of 1893. Klein was the representative of the Prussian government, and was authorized to explain these models. Out of the talks and conferences occasioned by them grew the Evanston Colloquium.

The next two papers, on the connectivity of surfaces, and on relations among the singularities of algebraic curves, follow minutely the same ideas as those developed in the memoir on cubic surfaces. The most important element is spatial intuition. Now follow four papers, together over eighty pages, on the combination of analysis and spatial intuition, that are entirely convincing, are strikingly beautiful, and seem curiously simple when approached from this point of view. They are on a new form of Riemann surfaces, and their application to the study of Abelian integrals of genus 3.

The method is to study a composite quartic curve, consisting of two conics, and fix the rational parameters of the points of contact of the common tangents. Then replace the conics by a non singular quartic. Now bitangents appear, and points of inflexion, the arrangements of which can be partly determined immediately from the figure. The rational parameters are replaced by Abelian integrals, and the various cases are so clearly separated that one wonders why the subject can be so hard.

Much later (1892) the same ideas were applied to the normal curve of genus p. While the results here obtained are both striking and important, it soon becomes evident that the last word on the uniformization of algebraic curves has not yet been said.

A short paper on the discriminant variety of an algebraic curve, also prepared for the Dyck catalogue, and a note on the geometric interpretation of the successive convergents of a continued fraction complete the list of contributions to the study of space relations. The next forty pages are devoted to the theory of knowledge and to the foundations of mathematics. The point of view is that we learn by experience and perfect our knowledge by successive approximations. In his 1901–02 course on the applications of mathematics Klein advocated a theory of errors for graphical processes analogous to that of least squares for numerical data obtained from observation. Astronomers require extreme numerical nicety based on crude analytic formulas, yet the sequel has shown that they were most frequently right. The same phenomenon is observed in connection with the singularities of plane curves. Both the original formulas of Plücker and the further relations established by Klein were first established empirically and later justified by analysis without the use of figures.

The first paper concerns the concept of a function, the result being that, so far as graphical or numerical control is concerned, we must deal with a band rather than with a mathematical curve. The substance of this paper, presented in 1873, was further developed in the Evanston Colloquium, and still further in the later paper on the arithmetization of mathematics. In the attempt to have this point of view crystalized the Beneke prize was offered in 1898, but was not awarded, as no memoir of required merit was submitted. This was partly due to the vagueness in the statement of the prize problem. The desired direction is indicated in the next paper, but the most significant statement in it is the old slogan "there is no royal road in mathematics." The last paper is a short address on the border questions of mathematics and philosophy delivered in Vienna in 1906. It emphasizes the difference between intuition and demonstration.

The second part, finite groups of linear substitutions, occupies 250 pages, and includes twelve memoirs, and some dozen pages of comments. The first of these memoirs was written in 1871, and the last in 1905. The first incentive to study the problems here considered is found in the short note written by Klein and Lie on those curves (W-curves) having the property that every tangent meets the sides of the triangle of reference in points which with the point of contact have a constant cross ratio. The thought is that curves invariant under finite groups of linear transformations have special properties, the study of which will contribute both to our knowledge of the curves and of the groups to which they belong. The use of projective geometry and of the geometry of inversion is fully justified by the results these subjects produced in the *Ikosaeder* and later *Modulfunktionen*. In particular, the linear transformations of the complex plane and their interpretation in terms of rotations of the regular polyhedra form the key note of the earlier papers. The important papers of Schwarz, making use of the 120 alternately congruent and symmetric triangles on the surface of the sphere, defined by the icosahedron, and later the classic on the algebraic solutions of the hypergeometric differential equation, were eagerly studied and compared with his own interpretations. The study received a big impetus when Gordan came to Erlangen in 1874; for several years he was one of the most eager workers in this field. Although Klein left Erlangen for the technical school of Munich in 1875, intimate association with Gordan was continued until 1880, when Klein moved to Leipzig.

During these years Brioschi and Kronecker were occupied with the quintic equation. On account of the similarity of the form of results in the two inquiries, it was suggested to Klein that knowledge of the solvability of the quintic equation would be of use to him in the study of the icosahedron, but he proposed the converse problem, namely, to make the icosahedron the basis of the study of the quintic equation. By recalling the works of these men now it is fairly evident that each received great assistance from all the others, and there is no question that the geometric interpretation was an important factor for all of them. The most important illustration probably was the discovery of the simple ternary linear group of order 168, in connection with the transformation of order seven of the elliptic function, and the quartic curve which belongs to the group. How incomparably more instructive is the study of this curve than to confine one's attention to the general quartic. Klein refers to this period as the happiest and most fruitful of his life. But his intense activity began to undermine his health, and when he assumed new duties in Leipzig a break had to come. After a prolonged leave of absence, Klein realized that he would be obliged to work under lower pressure, and contented himself with systematizing and arranging results already obtained, rather than continuing the former program. In this period the *Ikosaeder* was written. Much of its substance had already appeared in memoirs, but in condensed form, and accompanied by considerable discussion of invariants. In the book the presentation is much more elementary, and most of the considerations of the theory of invariants is omitted. Klein protests against the idea that his book is only a visualization of a theory worked out by others; he maintains rather that it contains indispensable elements of the theory itself. At any rate Gordan succeeded in simplifying materially his own presentation after the book appeared. But Gordan was also able to simplify Klein's work on the sextic (MATHEMATISCHE ANNALEN, vol. 68 (1909)). This fact is graciously mentioned by Klein, who adds that the simplest and most elegant presentation is that given by Coble (MATHEMATISCHE ANNALEN, vol. 70 (1911)).

The study of the equation of the sixth degree was given to pupils, in particular to Reichardt [*] and to Cole.[†] Klein had made various attempts to employ quaternary linear groups in the study of the sextic equation, and dramatically describes the general surprise when Wiman called attention (MATHEMATISCHE ANNALEN, vol. 47 (1896)) to the existence of a simple linear ternary group of order 360, which had been found by Valentiner in 1889. Since this was first published in Danish, it was apparently entirely unknown until mentioned by Wiman, who showed that it was simply isomorphic with the even permutations of six letters. That the general solution of the sextic could be made to depend upon this group rather than upon a quaternary group was later proven by Klein in a letter to Castelnuovo. This letter was presented to the Accademia dei Lincei by Castelnuovo, and published in its RENDICONTI (ser. 5, vol. 8 (1899)). It appears in the present volume as memoir LX, and occupies page 480.

The last part of the volume, pages 505–713, is devoted to mathematical physics and contains memoirs LXII to LXXX inclusive. There are two sub-headings, one being Linear Differential Equations and the other General Mechanics. Klein had repeatedly stated that when he was a student his main interest was in physics, and that he deliberately planned an activity in mathematics for some years, as part of his preparation for work as a physicist. Although this plan could not be realized to any great extent, still, when we rehearse Klein's influence in furthering the knowledge and application of mathematical physics, we must concede that it was no mean accomplishment.

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[*] *Ein Beitrag zur Theorie der Gleichungen sechsten Grades*, LEIPZIGER BERICHTE (1885), and *Ueber die Normierung der Borchardtschen Moduln der hyperelliptischen Funktionen vom Geschlecht* $p = 2$, MATHEMATISCHE ANNALEN, vol. 28 (1886).

[†] *A contribution to the theory of the general equation of the sixth degree*, AMERICAN JOURNAL, vol. 8 (1886).

We are told in the introduction that the first stimulus was the association with Neumann while at Leipzig—1880 to 1886. The first papers were those on Lamé functions, closely followed by the monograph on Abelian integrals, all containing many physical concepts.

During these years Klein made several trips to France and to England, and he emphasizes how his ideas of mechanics were greatly clarified and extended by them, particularly through Maxwell and Hamilton. He takes pride in having had three important English books translated into German, Routh's *Dynamics*, Lamé's *Hydrodynamics* and Love's *Elasticity*. Two other books, written at a later time, were direct outgrowths of his courses, and have been standard books in their fields ever since. These are Pockel's *Ueber die Differentialgleichung* $\Delta u + k^2 u = 0$ *und deren Auftreten in der mathematischen Physik* (1891), and Bôcher's *Ueber die Reihenentwicklungen der Potentialtheorie* (1894). After Schwarz was called to Berlin (1892), Klein commenced a comprehensive plan of enlarging and broadening the mathematical instruction at Göttingen. Weber was called in 1892, succeeded by Hilbert in 1895, who largely directed the work in pure mathematics, while Klein devoted his energy to filling in the gap between mathematics and physics, assisted by Brendel, Sommerfeld, Riecke, Runge, and others. At this time Klein was sent to the Chicago exposition, and returned with American ideas of financing his new institute. Through the assistance of Althoff, minister of public instruction, he received liberal appropriations from the government, and through the assistance of Althoff as a private citizen he also gained the interest and generous financial support of several men prominent in the industrial world. By these two means he built up an extensive and symmetric institute, comprising a large number of branches of mathematics, and holding from its origin the front rank in the mathematical world. The encyclopedia was started in 1894; Klein suggested the plan, and personally assumed the editorship of the part on mechanics, which is just now nearing completion.

During the next decade he was active in so many organizations, and participated in so many international undertakings that his own productivity as an investigator decidedly diminished, yet as representative of the universities in the Prussian senate, as chairman of the International Commission on the Teaching of Mathematics, as member of the commission on the international catalogue of the Royal Society, and finally as chairman of the union of German teachers of science he exercised a tremendous influence in shaping mathematical instruction. His constant aim was not merely passively to understand the principles, but so to apply them as to make for continued progress, to intensify and enlarge life. Although these activities were all in the interests of applied mathematics, yet Klein has continued the directorship and editorship of the MATHEMATISCHE ANNALEN since the death of Clebsch; he completed fifty years in this office last November.

The memoirs on Lamé functions are followed by those on the zeros of the hypergeometric series, the representation of the hypergeometric function by means of definite integrals, and the auto-reviews of the autographed lectures on the hypergeometric function and the linear differential equation of the second order, given in 1893–4. These were all published in the MATHEMATISCHE ANNALEN. The remaining essays were all published elsewhere. They include: a short report on recent English investigations on mechanics, a discussion of space collineations which occur in optical instruments, the greeting given at the opening of the mathematical congress at Chicago, the Princeton sesquicentennial lectures, two papers on graphical statics, one on Painlevé's criticism of Coulomb's law of friction, and finally one on the formation of vortices in frictionless liquids. The list is followed by a detailed explanation of the causes which led to the respective studies. The third and final volume is now in press. It contains the memoirs in the theory of functions.

VIRGIL SNYDER

LÉVY ON FUNCTIONALS

Leçons d'Analyse Fonctionnelle. By Paul Lévy, avec une préface de J. Hadamard. Paris, Gauthier-Villars, 1922. vi + 442 pp.

The increasing importance which is being given to the theory of functionals, or functions of lines, is illustrated by the fact that three of the Borel monographs in the last ten years have been concerned with this branch of mathematics, and the great breadth of the subject is illustrated by the fact that there is so little overlapping between the most recent of these, which is the subject of this review, and the earlier ones by Volterra,* and the more recent Cambridge Colloquium Lectures by Evans. In his introductory chapter, Lévy makes an interesting distinction between "algèbre fonctionnelle" and "analyse fonctionnelle." The first includes problems in which the unknowns are ordinary functions, but where the methods of the theory of functionals are used in determining them. The second includes problems where the unknowns themselves are functions of lines, or where the problems themselves could not be considered independently of the notion of a functional. Most of the work of Volterra and Evans mentioned above would belong to the "algèbre." The present monograph is primarily concerned with the "analyse."

The idea of a continuous functional is of fundamental importance. A functional $U(x(t))$ is said to be continuous if $U(y_n(t))$ approaches $U(x(t))$

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* *Leçons sur les Equations Intégrales et les Équations Intégro-differentielles*, reviewed by Westlund in this BULLETIN, vol. 20 (1914), pp. 259–62, and *Leçons sur les Fonctions des Lignes*, reviewed by Bliss, ibid., vol. 21 (1915), pp. 345–55.

as a limit when the "distance" between $y_n(t)$ and $x(t)$ approaches zero. Thus there are as many definitions of a continuous functional as there are definitions of the "distance" between two functions. He mentions several different definitions which have been used, and develops the theory based on two of them to some extent. The definition which he prefers is that the distance r between $x(t)$ and $y(t)$ is given by the formula

$$r^2 = \int_0^1 [y(t) - x(t)]^2 dt.$$

In other words $U(x(t))$ is continuous if $U(y_n(t))$ approaches $U(x(t))$ when $y_n(t)$ approaches $x(t)$ in the mean. Under this definition continuous functionals of the first degree, that is, linear functionals, can always be expressed as Lebesgue integrals, homogeneous functionals of the second degree as double integrals, etc. According to the other definition the distance is the least upper bound of $|y(t) - x(t)|$, and consequently $U(y_n(t))$ does not need to approach $U(x(t))$ unless the sequence $y_1(t)$, $y_2(t)$, $\cdots$ approaches $x(t)$ uniformly, in order to be continuous. This makes it necessary to introduce Stieltjes integrals instead of those of Lebesgue. The author discusses the consequences of this definition under the heading, "le point de vue logique," explaining clearly Hadamard's theorem that a linear functional is the limit of a sequence of definite integrals, F. Riesz' proof that it is also expressible as a Stieltjes integral, and some of Fréchet's work on bilinear functionals. He does not consider this point of view so useful in the applications as "le point de vue pratique," and in the last part of Part I and in the other two parts of the book, he uses the definition based on approach in the mean. The appropriateness of these designations of the two points of view may be questioned, particularly by those who believe that the Stieltjes integral, which is now being used to good advantage by a few applied mathematicians, is destined to be of much greater use in applied science. At the present time, however, the "point de vue logique" appeals particularly to those who are interested in generalizing a mathematical theory for its own sake.

In general Part I is written in a very interesting as well as elementary manner, and is a very valuable introduction to this branch of mathematics.

The second part is mostly the result of the author's own researches, although he refers frequently to the work of Gateaux. The first type of equations considered in Part II is a generalization of equations involving total differentials. The equation

$$du(x_1, x_2, \cdots, x_n) = \Sigma p_i(x_1, x_2, \cdots, x_n, u)dx_i$$

has for its analog

$$\delta U[x(t)] = \int_0^1 f[x(t);\ U, t]\delta x(t)dt,$$

and its conditions of integrability are similar. A system such as

$$\frac{\partial u(x_1, \cdots, x_n, y_1, \cdots, y_n)}{\partial x_i} = f_i\left(x_1, \cdots, x_n, y_1, \cdots, y_n, \frac{\partial u}{\partial y_1}, \cdots, \frac{\partial u}{\partial y_n}, u\right) \qquad (i = 1, 2, \cdots, n)$$

suggests the equation

$$U_x'(x(t), y(t);\ \tau) = f(x(t), y(t), U_y';\ U, \tau),$$

where U_x' is the functional derivative of U considered as a function of $x(t)$, with $y(t)$ entering as a parameter, and U_y' defined similarly. The notions of characteristics, and complete integral, and Cauchy's method are extended to such equations.

If $x(t)$ belongs to a very general class of functionals defined on (0, 1), it can be approached in the mean by a sequence of functions $x_1(t), x_2(t), \cdots$ such that $x_n(t)$ is constant in each of the intervals

$$\left(\frac{r-1}{n} < x \leqq \frac{i}{n}\right).$$

Consequently if $U(x(t))$ is continuous, $U(x_n(t)) \rightarrow U(x(t))$ as $n \rightarrow \infty$. If then x_i is the value of $x_n(t)$ in the ith of these intervals, $U(u_n(t))$ may be considered as a function of n variables and may be called $u_n(x_1, x_2, \cdots, x_n)$. Since

$$\int_0^1 x_n^2(t)dt = \frac{1}{n} \sum_{i=1}^{n} X_i^2,$$

every point inside the sphere in "function space" whose equation is

$$\int_0^1 x^2(t)dt = R^2 \tag{1}$$

can be said to be approached by a sequence of points in the respective n-dimensional spheres

$$\sum_{i=1}^{n} x_i^2 = nR^2, \tag{2}$$

and the $U(x(t))$ will be approached by a sequence of functions $u_n(x_1, \cdots, x_n)$. Also the mean value of $U(x(t))$ in the sphere (1) may be defined as the limit of the mean value of $u_n(x_1, \cdots, x_n)$ in the spheres (2).

Part III begins with a discussion of the sphere in n dimensions, whose radius is $R\sqrt{n}$. If two such spheres have radii $R(1-\alpha)$ and R, where α is an arbitrarily small positive number, the ratio of their volumes will be $(1-\alpha)^n$ which approaches zero as $n \rightarrow \infty$. Thus it may be stated that all the volume of a sphere in function space is located arbitrarily near its surface, excepting a part of relative measure zero. Similarly it is proved that the same volume is concentrated arbitrarily near the equator. It can be proved by means of Stirling's formula that the ratio of the volume of an n-dimensional sphere to the expression $(2\pi e)^{n/2}R^n/\sqrt{n\pi}$ approaches unity as $n \rightarrow \infty$. It follows that the volume approaches zero as $n \rightarrow \infty$ if $R \leqq 1/\sqrt{2\pi e}$ and otherwise becomes infinite. Thus the volume of a sphere in function space may either vanish or be infinite. This does not interfere with the determination of the mean of a functional defined over the sphere, however, as the mean of the corresponding function in n-space may approach a definite limit. The mean is used in generalizing Green's formula, and the equation of Laplace in function space is discussed, and various applications are made.

The preface by Hadamard emphasizes the importance of the subject, and the value of the author's contributions to it.

C. A. FISCHER

SHORTER NOTICES

L'Oeuvre scientifique de Laplace. By H. Andoyer. Paris, Payot & Cie., 1922. 162 pp.

After a brief sketch of the life of Laplace, Andoyer sets forth the characteristics of the works of this great French scientist. Andoyer cites the problems in celestial mechanics, which the eighteenth century mathematicians encountered, and reminds the reader how, through insufficient approximation, doubt was cast for a time upon the validity of Newton's law of inverse squares, and how a closer numerical approximation dispelled those doubts. Andoyer presents evidence showing the excessive harshness of the judgment passed upon Laplace by certain writers, to the effect that Laplace, in his writings, often failed to give due credit to his predecessors and contemporaries. Laplace's relations to D'Alembert, Biot and Poisson are described. Andoyer explains how Laplace again and again returned to certain topics in order that he might improve his exposition and perhaps free the subject from metaphysical entanglements. Not altogether surprising is Laplace's lack of interest in certain abstract fields of mathematics, like the theory of numbers. But strange is Laplace's adherence to Newton's corpuscular theory of light a quarter of a century after Thomas Young had advocated the undulatory theory and a decennium after Fresnel had won Arago over to the latter theory.

Andoyer's masterly account of Laplace's researches on celestial mechanics, on the figure of the earth, on the tides, on the système du monde, on the analytical theory of probability, and of researches on physics contains numerous quotations from the works of Laplace, bearing on points of scientific and philosophical interest. Andoyer's booklet will be enjoyed by students interested in the evolution of the mathematical sciences. An alphabetical index would have enhanced still further the usefulness of the book.

FLORIAN CAJORI

La Composition des Mathématiques dans l'Examen d'Admission à l'Ecole Polytechnique de 1901 à 1921. By F. Michel and M. Potron. Paris, Gauthier-Villars, 1922. 12 + 452 pp.

This volume contains the questions asked in the examinations for admission to the Ecole Polytechnique for the twenty years from 1901 to 1921, together with complete solutions. It should be of great value anywhere as a source book of interesting and rather difficult problems in the fields of analytic geometry, calculus and dynamics. It will be chiefly illuminating to American readers, however, as exhibiting the very high standard in force in mathematical instruction in France. Probably only a very small percentage of our college graduates who have specialized in mathematics would be able to make a creditable showing on any of these examinations. The book therefore presents to us in this country a high ideal towards which we may aspire, even though under present conditions it seems far from attainable.

J. W. YOUNG

Elements of Projective Geometry. By G. H. Ling, George Wentworth, and D. E. Smith. Boston, Ginn and Company, 1922. vi + 186 pp.

Although there is no date on the title page, nor in the preface, this book, according to the notice of the copyright by Ginn and Company, was apparently published in 1922. The omission of the date of publication, however, does not matter, since the contents and treatment are such that the *Elements* might have been written in Steiner's time. Indeed the topics and their presentation are in the main those of the early period of synthetic geometry, which is well known to critical students of projective geometry.

Within its elementary restricted domain the book is written clearly and concisely and may serve very well as a first introduction to a modern university course in projective geometry. The reviewer is nevertheless of the opinion that such an introductory and elementary treatise could be written with a view to its bearing upon the foundations of geometry and on the channels which lead to the fountainheads of contemporary geometric science. Even in an elementary course an occasional and opportune outlook into higher domains is invigorating and creates in the ambitious student a desire to penetrate deeper into the subject.

Welcome features of the book under review are the large number of exercises and a short chapter on the history of projective geometry. Clifford's term *cross ratio* seems to us preferable to Chasles' *anharmonic ratio.* Harmonic is a more or less mystic designation for $(ABCD) = -1$, and anharmonic is a negative definition for an infinite number of cross ratios which are not equal to -1. In the short historic sketch the important fact ought to be pointed out that von Staudt made the first successful attempt to establish the foundations of projective geometry (Geometrie der Lage) by introducing the concept of "Wurf" = throw of four collinear points independent of metric concepts (cross ratio of metric segments). After the remarkable developments by Poncelet, Steiner, Moebius, Chasles, etc., von Staudt's geometry of position was by far the most important step in the early critical development of projective geometry.

ARNOLD EMCH

Elementary Vector Analysis. By C. E. Weatherburn. London, G. Bell and Sons, 1921. xxvi + 184 pp.

The author's object in this book is to present the simpler portions of vector analysis and to apply them to portions of mechanics. He adheres to the notation of Gibbs. He gets as far as differentials and integrals, but does not bring in the notions of curl, convergence, and other ideas that belong to the general study of fields. The definitions are geometric for the scalar and the vector products, the vectors being always thought of as lines, or geometric vectors. A summary at the end of nineteen pages makes it very easy to find formulas and definitions. The geometry of curves in space is treated briefly, kinematics and dynamics of a particle, systems of particles, rigid kinematics, rigid dynamics, rigid statics.

The book should serve as a simple introduction to these subjects treated by way of the vector methods, and for the purposes in view is admirably adapted to the student's needs.

JAMES BYRNIE SHAW

Die elliptischen Funktionen und ihre Anwendungen. Zweiter Teil. By Robert Fricke. Leipzig, B. G. Teubner, 1922. viii + 546 pp.

Almost exactly 171 years ago Euler was led to the discovery of the addition theorem for elliptic integrals of the first kind (*Institutiones Calculi Integralis*) by the researches of Count Fagnano. Hence Jacobi designated the 23d of December, 1751, as the birthday of elliptic functions. During this long period of development the most fruitful subject has been the theory of the transformations of elliptic functions. Enriched and organized by related fields, in particular by the theory of groups, the subject of elliptic functions as it appears today forms one of the most beautiful edifices ever erected by the human intellect.

The second volume of Fricke's treatise is devoted to the addition theorem and to the transformation theory. It has an introductory chapter of 155 pages containing concepts, definitions, and theorems from the theory of groups, algebraic functions, algebraic numbers, etc. The first part (131 pp.) deals with the addition theorem and the related multiplication and division theorems. The remainder of the book is devoted to the transformation theory.

The first volume appeared in October, 1915, and was reviewed for this BULLETIN (vol. 23 (1917), p. 319). The method of presentation and the underlying thought of the three volumes was there stated in some detail.

Volume II appeared in 1922, and maintains the characteristic Klein-Fricke style. Much use is made of geometry in picturing the behavior of groups by means of their fundamental domains. The book is clearly printed on excellent paper, and there are very few errors.

L. WAYLAND DOWLING

Fluoreszenz und Phosphoreszenz im Lichte der neueren Atomtheorie. By Peter Pringsheim. Berlin, Julius Springer, 1921. 202 pp. 32 figs.

Although a few interesting analogies to the phenomena of fluorescence and phosphorescence exist in the theory of small vibrations, classical dynamical principles, applied to either the elastic solid or the electromagnetic theory of light, have never been successful in accounting for the observed effects. The reason for this is that these phenomena are dependent upon the emission and absorption of light, and, so far, only the newer quantum dynamics appears to offer the solution.

A comprehensive account of the important results in this field of research is contained in this small book by Pringsheim, together with indications of their interpretation with the aid of the quantum hypothesis. According to this, the effect of the incident light is to displace the electrons in the atoms from their normal orbits; on falling back into these the fluorescent light is emitted in accordance with Bohr's principle. No account is given of the related phenomena of thermo-luminescence, nor luminescence produced by other than light radiation.

Such a resumé of the newer work upon a fairly wide field of research is of very great value, particularly when done as well as this. A useful feature is a list of 266 articles bearing upon this subject that have appeared since 1908. For earlier work, reference should be made to the fourth volume of Kayser's *Handbuch der Spektroskopie*.

E. P. ADAMS

NOTES

The third number of volume 23 of the TRANSACTIONS OF THIS SOCIETY (April, 1922) contains the following papers: *Some generalizations of geodesics*, by E. J. Wilczynski; *On the gyroscope*, by W. F. Osgood; *The relative distribution of the real roots of a system of polynomials*, by C. F. Gummer; *A general theory of conjugate nets*, by E. P. Lane; *Parallel maps of surfaces*, by W. C. Graustein.

The Alfred Ackermann-Teubner prize of the University of Leipzig has been awarded to Professor Paul Koebe, for his three memoirs *Über die Uniformisierung der algebraischen Kurven*, published in volumes 67, 69, and 72 of the MATHEMATISCHE ANNALEN.

The Lasserre prize has been awarded by the minister of instruction of France to Professor Paul Langevin, of the Collège de France.

At Cambridge University, the Adams prize for this year has been awarded to Professor J. Proudman, of the University of Liverpool; the subject was *Tidal theory*. The subject announced for this prize for the period 1923–1924 is *The physical state of matter at high temperatures*. Smith's prizes have been awarded to Mr. J. C. Burkill, Trinity College, for an essay on *Functions of intervals and the problem of area*, and to Mr. A. E. Ingham, Trinity College, for an essay on *Mean value theorems in the theory of the Riemann zeta-function*. Rayleigh prizes have been awarded to Mr. E. F. Collingwood, Trinity College, for an essay on *The formal factorization of an integral function of finite integral order*, to Mr. W. R. Dean, Trinity College, for an essay on *The elastic stability of a plane plate*, to Mr. E. C. Francis, Peterhouse, for an essay on *The Denjoy-Stieltjes integral*, to Mr. C. G. F. James, Trinity College, for an essay on *The analytical representation of systems of space curves*, and to Mr. M. H. A. Newman, St. John's College, for an essay *On discontinuities of functions of a single real variable*.

Professor William Cain, of the University of North Carolina, has been awarded the J. James R. Croes medal of the American Society of Civil Engineers for his paper on *The circular arch under normal loads*.

Professor H. Ludendorf, of the Astrophysical Observatory at Potsdam has been elected member of the Berlin Academy of Sciences.

Professor Albert Einstein delivered a course of lectures at Madrid in March. On this occasion he was elected a member of the Madrid Academy of Science.

Professor Florian Cajori, of the University of California, has been elected Vice-president of Section L of the American Association for the Advancement of Science.

The following 19 doctorates with mathematics as major subject were conferred by American universities in the academic year 1921–1922; there is added in each case the month and year in which the degree was granted, the minor subject (or subjects) if outside of mathematics, and the title of the dissertation. C. R. Adams, Harvard, June 1922, *The general theory of the linear partial q-difference equation and of the linear partial difference equation of the intermediate type.* B. H. Brown, Harvard, June 1922, *The equilong transformations of Euclidean space.* Margaret Buchanan, Bryn Mawr, June 1922, Physics, *Systems of two linear integral equations with two parameters and symmetrisable kernels.* E. H. Carus, Chicago, December 1921, *Invariants as products: a vector interpretation of the symbolic method.* W. E. Cederberg, Wisconsin, June 1922, Astronomy, *On the solution of the differential equations of motion of a double pendulum.* H. S. Everett, Chicago, June 1922, *Determination of all general homogeneous polynomials expressible as determinants whose elements are homogeneous polynomials.* V. D. Gokhale, Chicago, March 1922, *Concerning compact Kürschák fields.* Claribel Kendall, Chicago, September 1921, *Certain congruences determined by a given surface.* R. E. Langer, Harvard, June 1922, I. *Developments associated with a boundary problem not linear in the parameter.* II. *The boundary problems and developments associated with a system of linear differential equations of the first order.* C. C. MacDuffee, Chicago, September 1921, *Invariantive characteristics of linear algebras with the associative law not assumed.* E. D. Meacham, Chicago, June 1922, *Properties of surfaces whose osculating ruled surfaces belong to linear complexes.* Eugenie M. Morenus, Columbia, June 1922, *Geometric properties completely characterizing the set of all the curves of constant pressure in a field of force.* Anna M. Mullikin, Pennsylvania, June 1922, *Certain theorems relating to plane connected point sets.* Eleanor Pairman, Radcliffe, June 1922, *Expansion theorems for solution of a Fredholm's linear homogeneous integral equation of the second kind with kernel of special non-symmetric type.* H. P. Pettit, Illinois, June 1922, Physics and Statistics, *A general cyclide with special reference to the quintic cyclide.* J. F. Reilly, Iowa, July 1921, Physics, *On certain generalizations of osculatory interpolation.* W. P. Udinski, Illinois, June 1922, Chemistry and Electrical Engineering, *On a series of rational functions formally analogous to Fourier's series.* B. C. Wong, California, May 1922, Astronomy, *A study and classification of ruled quartic surfaces by means of a point-to-line transformation.* E. B. Zeisler, Chicago, June 1922, Astronomy, *Definite integral representation of invariants.*

The following five doctorates, conferred in the latter half of the calendar year 1922, are listed in the same way as the preceding. E. H. Clarke, Chicago, September 1922, Astronomy, *On the minimum of the sum of a definite integral and a function of a point.* J. D. Eshleman, Chicago, September 1922, *The Lagrange problem in parametric form in the calculus of variations.* W. S. Kimball, Chicago, September 1922, Physics, *Scattering of particles by an Einstein curve.* Harry Langman, Columbia, November 1922, *Conformal transformations of period n and groups generated by them.* J. S. Turner, Chicago, *Fundamental system of formal invariants of a modular group of transformations.*

The following four doctorates in mathematical physics or mathematical astronomy were conferred during the calendar year 1922; there is added in each case the month in which the degree was conferred, the major subject, the minor subject, and the title of the dissertation. F. E. Carr, Chicago, December, mathematical astronomy, pure mathematics, *A solution of the problem of two bodies one of which is a rotating oblate spheroid.* H. C. Levinson, Chicago, September, mathematical astronomy, pure mathematics, *The gravitational field of masses relatively at rest according to Einstein's theory of gravitation.* Louis Slichter, Wisconsin, June, mathematical physics, applied mathematics, *An experimental study of an acoustic system.* Warren Weaver, Wisconsin, June, mathematical physics, applied mathematics, *A summary of the analytic formulation of the theory of electrodynamics.*

Professor K. Heun, of the Karlsruhe Technical School, has retired from active teaching.

Dr. P. Finsler has been admitted as privat docent at the University of Cologne.

Cambridge University has invited Professor H. A. Lorentz, of the University of Haarlem, to deliver the Rede Lecture, on May 15; the subject will be *Maxwell's electromagnetic theory.* Mr. A. Hutchinson, of Pembroke College, has been appointed university lecturer in crystallography.

At New Hampshire State College, Mr. W. E. Wilbur, Mr. Hubert Huntley, and Mr. P. F. Howard have been appointed instructors in mathematics. Professor H. I. Slobin, head of the department of mathematics, has been reappointed Director of the Summer School for the 1923 session.

Assistant Professor G. W. Mullins, of Barnard College, Columbia University, has been promoted to an associate professorship of mathematics.

Mr. W. F. Shields has been appointed professor of freshman mathematics at Fordham University. He succeeds Rev. L. J. McGarry, who has gone to Woodstock College, Maryland.

Assistant Professor F. N. Bryant, of Syracuse University, has been promoted to a full professorship of mathematics.

Dr. L. L. Smail, assistant professor of mathematics at the University of Washington, has been appointed assistant professor of mathematics at the University of Oregon, effective October 1, 1923.

Professor Max Abraham, formerly professor of mechanics at the Milan Technical School, died November 16, 1922, at Munich, at the age of forty-seven years.

Dr. Fritz Cohn, professor of theoretical astronomy at the University of Berlin, and director of the Berlin Rechen-Institut, died December 14, 1922, at the age of fifty-seven years.

Dr. James Gow, formerly headmaster of Westminster School, and author of *A short History of Greek Mathematics*, died February 16, 1923.

Mr. Rawdon Levett, one of the founders of the Association for the Improvement of Geometric Teaching, later the (British) Mathematical Association, and its first secretary, died February 1, 1923.

NEW PUBLICATIONS

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ASSIER DE POMPIGNAN (—.). Note sur le calcul tensoriel. Paris, Hermann, 1923. 8vo. 32 pp.

BOUCHARD (J.). Tables centésimales pour le tracé des courbes. Paris, Gauthier-Villars, 1922. 8vo. 204 pp.

BOUTROUX (P.). Les mathématiques. Paris, Albin Michel, 1922. 16mo. 184 pp.

BRYAN (G. H.). Mathematical tables. London, Macmillan, 1922. 28 pp.

EDWARDS (J.). A treatise on the integral calculus. Volume 2. London, Macmillan, 1922.

FERMAT (P.). Oeuvres de Fermat, publiées par les soins de Paul Tannery et Charles Henry: Documents inédits, publiés avec notices sur les nouveaux manuscrits par C. de Waard. Paris, Gauthier-Villars, 1922. 4to. 216 pp.

HENRY (A.). Calculus and probability for actuarial students. Published by the authority and on behalf of the Institute of Actuaries. London, C. and E. Layton, 1922. 7 + 152 pp.

HENRY (C.). See FERMAT (P.).

KRICHEWSKY (S.). A method of curve fitting. Cairo, Government Press, 1922. 4to. 17 pp.

LASLEY (J. W.). Some special cases of the flecnode transformation of ruled surfaces. (Diss., Chicago.) Chicago, 1922. 19 pp.

LORING (F. H.). Definition of equivalence. London, H. O. Lloyd, 1922. 16 pp.

ROMANE (R.). Géométrie du sens commun. Mulhouse, Brinkmann, 1922.

ROSNY (J. H.). Les sciences et le pluralisme. Paris, Alcan, 1922. 4 + 219 pp.

STUYVAERT (M.). Les nombres positifs. Manuel d'arithmétique. 3e édition. Gand, Van Rysselberghe et Rombaut, 1923. 8vo. 185 pp.

TANNERY (P.). See FERMAT (P.).

THOMPSON (S. P.). Calculus made easy. 2d edition, enlarged. London, Macmillan, 1921. 8vo. 301 pp.

VALENTINER (S.). Vektoranalysis. 3te, umgearbeitete Auflage. (Sammlung Göschen.) Berlin, Vereinigung wissenschaftlicher Verleger, 1923. 132 pp.

DE WAARD (C.). See FERMAT (P.).

WINGER (R. M.). An introduction to projective geometry. Boston, Heath, 1923. 14 + 443 pp.

WOODHOUSE (T.). Main counts and calculations. (Oxford Technical Manuals.) London, H. Frowde, and Hodder and Stoughton, 1921. 8 + 119 pp.

ZENON. Approximation dans l'interpolation. Paris, Gauthier-Villars, 1922. 8vo. 10 pp.

PART II. APPLIED MATHEMATICS

ABRAHAM (M.). Theorie der Elektrizität. Band 1: Einführung in die Maxwellsche Theorie der Elektrizität, von A. Föppl. 6te, umgearbeitete Auflage. Leipzig, Teubner, 1921. 8 + 389 pp.

ANDOYER (H.). Cours d'astronomie. 1re partie: Astronomie théorique. 3e édition, entièrement refondue. Paris, Hermann, 1923. 8vo. 455 pp.

ANDRADE (J.). Les organes réglants des chronomètres. Besançon et Bienne, E. Magron, 1922. 12mo. 146 pp.

D'AQUINO (L.). Il fondamento, il contenuto, le consequenze della nuova teoria della relatività. Esposizione critica elementare. Napoli, Detken e Rocholl, 1922. 8vo. 18 pp.

BATSON (R. G.) and HYDE (J. H.). Mechanical testing. Volume 1: Testing of materials of construction. London, Chapman and Hall, 1922. 13 + 413 pp.

BAXTER (H. E.). See YOUNG (G.).

BEHREND (B. A.). The induction motor and other alternating current motors. 2d edition. New York, McGraw-Hill, 1921. 23 + 272 pp.

BELLENOT (H.). See SOMMERFELD (A.).

BENRATH (A.). Physikalische Chemie. Heidelberg, Ehrig, 1922. 88 pp.

BIGOURDAN (G.). Mélanges scientifiques. Paris, Gauthier-Villars, 1922.

BRISSON (B.). See MONGE (G.).

CARTAN (E.). Sur les équations de la gravitation d'Einstein. Paris, Gauthier-Villars, 1922. 4to. 60 pp

CRAWFORD (W. J.). Elementary graphic statics. 2d edition. London, C. Griffin, 1923. 8vo. 140 pp.

DANVILLIER (A.). See LEDOUX-LEBARD (R.).

DE DONDER (T.). Premiers compléments de la gravifique einsteinienne. Paris, Gauthier-Villars, 1922. 4to. 38 pp.

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DA FANO (G.). Aufgaben aus der darstellenden Geometrie. 2te Auflage. Leipzig und Zürich, Speidel und Wurzel, 1921. 66 pp.

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Ledoux-Lebard (R.) et Danvillier (A.). La physique des rayons X. Paris, Gauthier-Villars, 1921. 7 + 441 pp.

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Maillard (L.). Quand la lumière fut . . . Tome I: Les cosmogonies anciennes. Paris, Les Presses Universitaires de France, 1922. 8vo. 214 pp.

Margoulis (W.). Les helicoptères. Recherches expérimentales sur le fonctionnement le plus général des hélices. Etudes sur la mécanique de l'hélicoptère. Paris, Gauthier-Villars, 1922. 11 + 90 pp.

Monge (G.). Géométrie descriptive. Augmentée d'une théorie des ombres et de la perspective extraite des papiers de l'auteur par Barnabé Brisson. I–II. (Les Maîtres de la Pensée Scientifique.) Paris, Gauthier-Villars, 1922. 16 + 144 + 138 pp.

Müller (A.). Die philosophischen Probleme der Einsteinschen Relativitätstheorie. Braunschweig, Vieweg, 1922. 8 + 224 pp.

Niceforo (A.). Scopi e applicazioni del metodo statistico. Palermo, 1922.

Pender (H.). Direct-current machinery. A text-book on the theory and performance of generators and motors. New York, Wiley, 1923. 10 + 314 pp.

Réveille (J.). Dynamique des solides. Gyroscopes. Paris, Baillière, 1923. 8vo. 506 pp.

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Willis (E. J.). The mathematics of navigation. Richmond, Va., Fergusson, 1921. 8vo. 34 pp.

Winternitz (—.). Relativitätstheorie und Erkenntnislehre. Leipzig, Teubner, 1922.

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INCORPORATION OF THE AMERICAN MATHEMATICAL SOCIETY

At the February, 1921, meeting of the Council, Professors H. W. Tyler, W. F. Osgood, and D. E. Smith were appointed a committee with instructions to incorporate the Society and to procure a national charter if that were feasible. A bill to incorporate in the District of Columbia was introduced into the Senate and the House of Representatives, but it met with considerable opposition in the committee of the Senate. At the end of the last session of Congress the committee decided to abandon the idea of a special bill and to incorporate under the Code of the District of Columbia.

On May 3, 1923, incorporation was effected under the Code of the District of Columbia, the incorporators being H. W. Tyler, H. L. Hodgkins, Harry English, O. S. Adams, and J. T. Erwin.

The certificate of incorporation is very simple and contains beside the preamble only the following four articles.

> I. The name or title by which this Society shall be known in law is the AMERICAN MATHEMATICAL SOCIETY.
>
> II. The term for which it is organized shall be perpetual.
>
> III. The particular business and objects of the Society are the furtherance of the interests of mathematical scholarship and research.
>
> In furtherance of and not in limitation of the general objects of the Society and the powers conferred by the laws of the District of Columbia, it is hereby expressly provided that the following persons shall be eligible for membership in this corporation, to wit: the members now forming the unincorporated association of the same name and such other persons as may from time to time be elected to membership under the by-laws to be hereafter enacted. The corporation hereby constituted shall have power to fill all vacancies created by death, resignation, or otherwise, to provide for the election of foreign, domestic, or honorary associate members, and the division of such members into classes, and to make all needed rules and regulations for the purpose of carrying out its proper corporate objects.
>
> IV. The number of its trustees, directors or managers for the first year of this corporation's existence shall be thirty-one (31).

The first meeting of the Corporation was held at the Cosmos Club, Washington, D. C., on May 4, 1923, at 12:45 p. m. All the incorporators were present and they elected as their associates for the first year thirty-one members of the Council for 1921. The next meeting of the Corporation will take place at Poughkeepsie in connection with the summer meeting. It is expected that at that time the incorporated body will assume control of the affairs of the Society and continue the objects for which the old Society was organized.

R. G. D. RICHARDSON,
Secretary.

REPORT ON CURVES TRACED ON ALGEBRAIC SURFACES*

BY S. LEFSCHETZ

1. *Introduction.* An extensive literature to which we propose to devote this report has grown up around the following question and the related transcendental theory. *Given an algebraic surface, what can be said in regard to the distribution of its continuous systems of algebraic curves?*† For various reasons we have chosen in place of the chronological presentation one in which function theory and analysis situs play the predominant part, and that has been made possible by two papers of Poincaré (*s*, I, II). We must however recall at the outset that the general answer to the above question had been given earlier by Severi (*u*, V, VI), his methods being largely of an algebro-geometric nature (see § 14), except for the use of a very important transcendental theorem due to Picard (*q*, II, p. 241). In favor of the methods which dominate this report, it must be stated that they alone made possible the solution of some important problems, and furthermore have notably enriched the theory.

A question similar to the above may be asked concerning algebraic varieties, but in order to remain within proper bounds, we have deemed it best to omit them altogether.

2. *Connectivity Indices.* We shall have occasion to consider throughout a basic n-dimensional manifold W_n,‡ usually an algebraic curve ($n = 2$), or a surface ($n = 4$). A sum of closed, k-dimensional, two-sided, analytic manifolds in W_n is called a *k-cycle* of the manifold, and shall be denoted by Γ_k. If it bounds, it is a zero-cycle; the fact being expressed by a homology: $\Gamma_k \sim 0 \bmod W_n$. Homologies can be added

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* Presented before the Society at the Symposium held in Chicago, April 13, 1923.

† The properties of linear and continuous systems, so successfully investigated by Castelnuovo, Enriques, and Severi, have been described by them in three readily accessible reports (*q*, II, p. 485; *e*; *u*, XI). These and similar references refer to the bibliography at the end of the report.

‡ For a more extended topological discussion see *n*, II, § 1; *v*, Ch. 4.

and also multiplied (but not divided) by integers. If several k-cycles satisfy no homology they are *independent.* The maximum number R_k of such cycles is the kth *connectivity index* (*linear index* for $k = 1$). The index R_1 of an algebraic variety is always even. For an algebraic curve, $\frac{1}{2}R_1 = p$ is the *genus,* for an algebraic surface or variety, $\frac{1}{2}R_1 = q$ is the *irregularity* (n, II, p. 233; n, VII, Ch. 2).

3. *Abelian Integrals.* We recall that by an *Abelian integral* attached to the algebraic curve C, of order m and genus p, whose equation is $f(x, y) = 0$, is meant an integral

$$\int R(x, y)dx,$$

where R is rational in the coordinates of a variable point on C. A *period* of it is its value taken over a cycle Γ_1 of C. The integral is of the *first kind* when it is holomorphic everywhere on the curve. It is then of the form

$$\int \frac{Q(x, y)dx}{f_y'},$$

where Q is an adjoint polynomial of order $m - 3$.* The maximum number of linearly independent † integrals of the first kind is equal to p (see r). Let $u_1, u_2, \cdots, u_p$ be a set of such integrals, and consider the equations

$$\sum_{h=1}^{n} \int_{A}^{A_h} du_i = v_i, \qquad (i = 1, 2, \cdots, p),$$

where the v's are constant, and the unknowns are the upper limits A_h on C.

(a) When there is more than one solution there are an infinite number, and the most general one depends linearly upon $r \geqq n - p$ parameters (Abel; Clebsch; c, p. 395). Their totality constitutes a so-called *linear series* on C.

(b) For $n = p$ the solution is in general *unique* (inversion in the sense of Jacobi; see c, p. 463). In the exceptional case, a certain number $p - p'$ of the points may be assigned arbitrarily, and the remaining p' are then uniquely determined.

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* When C has no other singularities than double points with distinct tangents, an adjoint polynomial is one which vanishes at the double points. A similar definition holds for surfaces, with the double curve in place of the double points. See r.

† In the sense that no linear combination of them is constant.

4. *Algebraic Surfaces.* We shall denote our surface by F, take for its equation $f(x, y, z) = 0$ and assume it irreducible, of order m, with the genus of a generic plane section (to be denoted by H) equal to p. The special sections cut out by planes $y = Ct$ shall be called H_y. We shall furthermore assume F with only ordinary singularities* (double curve with triple points in finite number) and in arbitrary position as to the axes and the plane at infinity. The section H_y will then be of genus p unless its plane is tangent to F, when the genus is lowered to $p - 1$. The values $a_1, a_2, \cdots, a_n$ corresponding to contact are its *critical* values and n is the class of F. To each a_k is attached an important cycle Γ_1 of F: it is the cycle δ_k which tends to a point when y approaches a_k (vanishing cycle). When y turns around a_k, the increment of any Γ_1 of H_y is homologous to a multiple of δ_k mod. H_y (Picard, q, I, p. 95; n, VII, Ch. 2).

5. *Abelian Sums attached to Algebraic Curves.* A notable achievement it was of Poincaré's to have discovered a set of simple expressions that characterize continuous systems of algebraic curves on F (s, II, p. 56). By taking advantage of certain results known before his work but which follow readily also from his methods, we shall obtain easily a set of somewhat simpler expressions that will suffice for our purposes (n, IV, p. 343; u, IX, p. 204).

To H_y are attached p linearly independent integrals of the first kind of the particularly simple type

$$u_j = \int \frac{Q_j(x, y, z)}{f_z'}\, dx \qquad (j = 1, 2, \cdots, p),$$

where the Q's are adjoint polynomials of order $m - 3$ in x and z; however q of them, say the last q, are of degree $m - 2$ in x, y, and z, while the remaining $p - q$ of them are still of degree $m - 3$ when y is taken into consideration. Furthermore the periods of u_{p-q+l} with respect to the vanishing cycles δ_k are all zero.†

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* Any F can be birationally reduced to that particular kind (Beppo Levi, o; Chisini, f).

† The integrals u_{p-q+l} are what q linearly independent integrals of total

Let now $A_1, A_2, \cdots, A_m$ be the fixed points of H_y (all at infinity), and let C be an arbitrary algebraic curve on F. The curve C may go through some A's with varied multiplicities. Let $B_1, B_2, \cdots, B_\nu$ be a group of points on H_y, including all variable points of intersection of C and H_y, and some of the A's, each taken with any multiplicity whatever. Let us then consider the sums

$$v_j(y) = \sum_{k=1}^{\nu} \int_{A_1}^{B_h} du_j, \qquad (j = 1, 2, \cdots, p). \tag{1}$$

From the behavior of the v's, we find the relations

$$\begin{cases} v_j(y) = \sum \dfrac{\lambda_k}{2\pi i} \displaystyle\int_{A}^{A_k} \dfrac{\Omega_{jk}(Y)dY}{Y - y}, & (j = 1, 2, \cdots, p - q), \\ v_{p-q+l}(y) = \alpha_l, & (l = 1, 2, \cdots, q), \end{cases} \tag{2}$$

where the α's are constants, the λ's are integers, and Ω_{jk} is the period of u_j with respect to δ_k. These expressions are basic for what follows.

6. *Existence Theorem and its Interpretation. Under what circumstances can the v's given by* (2) *yield, by means of* (1), *points B_h with an algebraic curve C for locus?* Assign to all but $p' \geqq p$ of the B's the position A_1, and solve for the remainder as in § 3, p' having the same meaning as there. On discussing the equations obtained (Poincaré, *s*, II, p. 75; *s*, III, p. 41; Lefschetz, *n*, VII, Ch. 4; Severi, *u*, IX, p. 278), the following necessary and sufficient conditions are obtained:

(*a*) The v's must be regular at $y = a$.

(*b*) Let there be constants β_h such that

$$\frac{1}{y - y_0} \sum \beta_h Q_h(x, y, z) \equiv Q(x, y, z) \tag{3}$$

is finite when y approaches y_0. Similarly, the expression

$$\frac{1}{y - y_0} \sum \beta_h v_h(y)$$

must equally remain finite when y approaches y_0.

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differentials of the first kind of F (see § 11) become when y is held constant. The possibility of subdividing the u's into two groups as here indicated is based upon the fundamental result recalled (loc. cit.), together with a theorem proved by Picard (*q*, II, p. 437) and Severi (*u*, VIII).

Conditions (a) and (b) are unsatisfactory notably in that they involve the particular variable y. From them may be derived others of a more significant nature and in particular not involving y (Lefschetz, n, III; n, IV, p. 345; n, VII, Ch. 4). The polynomial Q in (3) is adjoint of order $m-4$, hence

$$\int\int \frac{Q(x, y, z)}{f_z'}\,dxdy \tag{4}$$

is finite everywhere on F. In the usual terminology we say that it is of the first kind (Noether, p, I). As a matter of fact there is a relation (3) for every Q, hence (4) is arbitrary of its kind. We then have the following theorem.

In order that there may correspond to the (v)*'s an algebraic curve* C, *there must exist a cycle* Γ_2 *with respect to which the period of every double integral of the first kind is zero.**

The cycle Γ_2 is readily described. Let Δ_k be the locus of δ_k as y describes aa_k. Picard (q, II, p. 335) and Poincaré (s, II, p. 57) for finite cycles and Lefschetz (n, IV, Ch. 3) for any cycle, have established the homology

$$\Gamma_2 \sim \sum \lambda_k \Delta_k + \text{part of } H_a, \text{ mod } F.$$

When C exists we have (n, IV, Ch. 4)

$$C \sim \sum \lambda_k \Delta_k + \text{part of } H_a, \text{ mod } F,$$

the λ's being the same integers as in (2). This brings out clearly the relation between them and the curve.

7. *Equivalence of Curves.* A variable curve C of intersection of F with a surface whose equation contains r linear parameters gives rise to a *linear system of dimension* r, denoted by $|C|$. This system is *complete* if its curves do not belong to another of dimension $> r$, as shall be assumed throughout. The complete system determined by a given curve is unique.

If the irregularity q is greater than zero, the surface contains continuous systems whose curves do not all belong to one and the same linear system (e, p. 707). We denote by

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* A noteworthy feature of this theorem is that it is the only one known concerning periods of double integrals of the first kind. This is in striking contrast with Abelian integrals of the first kind for the periods of which an extensive body of theorems has long been available.

$\{C\}$ the amplest system generated by all variable curves that may approach C. It contains all systems $|C|$ determined by its curves, and may be thought of as an algebraic variety whose elements are linear systems. Indeed when $\{C\}$ is sufficiently general, this variety is an Abelian variety of genus q, the so-called Picard variety of F.

By $|C + D|$ and $\{C + D\}$ are meant the systems defined by the composite curve $C + D$. They are uniquely determined by $|C|$ and $|D|$ in the first case, and by $\{C\}$ and $\{D\}$ in the second case. If D is in $|C|$ (or in $\{C\}$) we write $|2C|$ (or $\{2C\}$) for the sum. The meaning of $|kC|$ and of $\{kC\}$ follows immediately, if k is a positive integer. Likewise the meaning of $|C - D|$ and of $\{C - D\}$ is immediate (see h). The last two systems may not actually exist; they are then called *virtual.** Although not represented by any geometric configuration, there are important related symbols, and the introduction of these systems (Severi, u, X) has been very useful. Similar remarks hold for $|k_1C_1 + k_2C_2 + \cdots + k_\nu C_\nu|$ and $\{k_1C_1 + k_2C_2 + \cdots + k_\nu C_\nu\}$, where the k's are integers.

The v's are the same for all curves of a linear system; and for curves of a continuous system, they differ in the constants α, but not in the integers λ. In any case it is clear that the v's of the same index are combined like their curves.†

A partial result of Poincaré's (s, II, p. 98) which I have completed is the following theorem (n, IV, Ch. 4).

If C and D have the same v's, then D belongs to $|C|$. If their integers λ are the same, i.e., if only the first $p - q$ functions v are the same, then there is a positive integer k such that $\{C + kH\}$ and $\{D + kH\}$ coincide.

This justifies the following definition: C and D are *equivalent*, and we write $C = D$ if there is an E such that $\{C + E\}$ and $\{D + E\}$ coincide. A meaning is then readily attributed to a relation

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* Moreover, even when $\{C - D\}$ exists it is not necessarily unique.

† Henceforth we mean by the set of v's corresponding to C that obtained when all points of intersection of C with H_y are considered and no others.

$$\mu_1 C_1 + \mu_2 C_2 + \cdots + \mu_\nu C_\nu = 0.$$

These notions go back essentially to Severi (*u*, V, p. 198). However, his definition of equivalence is somewhat narrow. The question has been searchingly examined by Albanese and our definition corresponds to what he calls *virtual equivalence* (*b*, p. 165).

8. *Identity with Homology.* Much interest is added to what precedes by this proposition (Lefschetz, *n*, IV, p. 347; *n*, VII, Ch. 4): *There is complete equivalence between the two relations*

$$\sum \mu_i C_i = 0; \qquad \sum \mu_i C_i \sim 0, \text{ mod } F.$$

Thus a purely algebro-geometric notion is reduced to one of analysis situs. Well known propositions from the latter theory yield at once these results, whose geometric content we owe to Severi (*n*, IV; *n*, VI; *u*, V; *u*, VI):

(*a*) *There is a positive integer* ρ *such that any* $\rho + 1$ *curves satisfy a relation of equivalence, this not being the case for some sets of* ρ *of them.* The number ρ is the well known Picard number (*q*, II, p. 241).

(*b*) *The number* ρ *is the maximum number of distinct two-cycles with respect to which every double integral of the first kind has a zero period.*

(*c*) *Given a curve* C, *of sufficiently general type,* there may exist* $\sigma - 1$ *others,* $C_2, C_3, \cdots, C_\sigma$, *such that* $C_i \neq C_1, \lambda C_i = \lambda C_1, \lambda > 1$. The number σ is the product of Poincaré's so-called *torsion coefficients* for linear or two-cycles † (they are the same for F).

(*d*) *There exists an ordinary base for the curves of* F, *i.e., a set of curves* $C_1, C_2, \cdots, C_\rho$, *such that whatever* C *we may write*:

$$\mu C = \mu_1 C_1 + \mu_2 C_2 + \cdots + \mu_\rho C_\rho.$$

The term *ordinary* base was introduced by Severi. By increasing the number of curves to $\rho + \sigma - 1$, we may have what Severi terms a *minimum base,* i.e., such that $\mu = 1$ for every C. Let these curves be $D_1, D_2, \cdots, D_{\rho+\sigma-1}$. Then

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* If virtual curves are admitted no restrictions are needed.

† Examples of surfaces with $\sigma > 1$ have been given by Severi (*u*, VI), Godeaux (*k*) and the writer (*n*, IV, p. 362). On such an F can be found a Γ_1 such that $\lambda\Gamma_1$ bounds ($\lambda > 1$) while Γ_1 does not, and similarly for two-cycles.

$$C = \nu_1 D_1 + \nu_2 D_2 + \cdots + \nu_{\rho+\sigma-1} D_{\rho+\sigma-1}.$$

All these curves may be actually chosen *effective* (non-virtual) except for $\rho = 1$, when it may be necessary to take $\sigma + 1$ curves in the minimum base, if it be desired to have them all effective. The geometric content of this answers completely the question proposed at the beginning of our report. Indeed, let $D_1, D_2, \cdots, D_\nu$ be a minimum base composed exclusively of effective curves. For every C we may write

$$C = \sum \mu_i D_i = \sum \mu_i'' D_i - \sum \mu_i' D_i,$$

where the μ''s and μ'''s are non-negative integers. Then there exists a positive integer μ such that the continuous systems

$$\{C + \mu H + \sum \mu_i' D_i\}, \qquad \{\mu H + \sum \mu_i'' D_i\}$$

coincide (Albanese, *b*, p. 204).

9. *Integrals of the Second and Third Kinds.* Abelian integrals constitute the natural analytical tool in the study of sets of finite points on an algebraic curve, for these points appear either as the set of singular points of the integral, or as the zeros of the integrand. For an entirely similar reason, the integrals that generalize Abelian integrals are of paramount interest in investigations on algebraic curves of a surface, and thus find a proper place here. We recall that an Abelian integral of the *second kind* is one behaving everywhere on its curve C like a rational function. Finally an integral is of the *third kind* if not of one of the other two kinds. Its singularities other than poles consist in a finite number of logarithmic points and to each belongs a logarithmic period corresponding to a small circuit surrounding the point. The sum of the logarithmic periods is zero. Hence there must be at least two logarithmic points, and in fact there is an integral having any two points of C for logarithmic singularities and no other. Integrals of the second kind are linearly independent if no linear combination of them reduces to a rational function. The maximum number of such integrals is twice the genus of C (see *r*).

10. *Generalization to Double Integrals.* On passing to F there are two modes of extending Abelian integrals, first to double integrals, next and much less obviously to integrals of total differentials (Picard, q, I, p. 102), which we shall consider below.

We have defined double integrals of the first kind in § 6. On the transcendental side there is no other theorem than the one mentioned there, although the number of linearly independent integrals and the related linear system cut out by adjoint polynomials of order $m-4$ (canonical system) have played a large part in investigations on surfaces (e, p. 704), this being due of course to their invariance under birational transformations. A question which I have been able to settle only in some special cases (n, IV, p. 349) is still outstanding: *Can a double integral of the first kind be without periods?** The answer (which is probably negative) would have an important bearing upon our subject.

In the several existing treatments of double integrals of the second kind (Picard, q, II; Lefschetz, n, II, p. 242; n, VII, Note I), the type

$$\int\int\left(\frac{\partial U}{\partial x}+\frac{\partial V}{\partial y}\right)dxdy\,, \qquad (U,V \text{ rational}), \tag{5}$$

introduced by Picard, plays somewhat the same part as that played by rational functions for Abelian integrals. The reason is here again invariance under birational transformations. The following mode of attack (n, VII, Note I) seems shortest and best, especially in that it exhibits a noteworthy theorem and is readily extended to integrals of any multiplicity of a higher variety: We take two-cycles in maximum number, say $\Gamma_2^1, \Gamma_2^2, \cdots, \Gamma_2^{\rho_0}$ such that no matter how ample a set of curves $D_1, D_2, \cdots, D_r$ is given, there is a cycle $\sim \Gamma_2^i$ not intersecting any of them. The curves $C_1, C_2, \cdots, C_\rho$ of a Severi base are two-cycles independent of the Γ_2^i, and every Γ_2 depends upon the $\rho_0+\rho$ thus obtained, whence

$$\rho_0 = R_2 - \rho.$$

It is then found that J is without period with respect to $\Gamma_2{}^i$, while on the contrary any integral *not* of this type will have a period with respect to some such cycle, whence it follows that the maximum number of linearly independent integrals of the second kind is ρ_0. But it is known (Alexander, *a*; Lefschetz, *n*, II, p. 239; *n*, VII, Ch. 3) that

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* It will be recalled that the corresponding theorem for Abelian integrals is proved with ease (see r).

$$R_2 = I + 4q + 2,$$

where $I = n - m - 4p$ is the well known Zeuthen-Segre invariant. From this follows Picard's formula

$$\rho_0 = I + 4q - \rho + 2.$$

The very suggestive relation between the number of linearly independent integrals of the second kind and the number of cycles of a certain type may be extended to multiple integrals of higher varieties (*n*, VI).*

11. *Integrals of Total Differentials.* By *integral of total differentials* is meant one of type

$$\int R dx + S dy, \qquad \frac{\partial R}{\partial y} = \frac{\partial S}{\partial x} \cdot$$

The classification and independence theorems are as for Abelian integrals with q in place of p (Picard, *q*, I, Ch. V; Castelnuovo-Enriques and Severi, *e*, p. 715; Poincaré, *s*, II, p. 91). The extension of Abel's theorem has been the object of extensive investigations by Severi (*u*, II, III).

We are particularly interested in integrals of the third kind. Let J be one, $C_1, C_2, \cdots, C_k$ its singular curves. In the vicinity of C_i the integral behaves either like a rational function or like a logarithm. In the latter case there is a logarithmic period, and C_i is a logarithmic curve. By investigating the integral which J determines on H_y, Picard has shown that when the logarithmic curves are arbitrary, J exists, provided that their number exceeds a certain integer ρ whose first appearance in the literature was precisely in this connection (*q*, II, p. 240). That it coincides with the integer denoted by ρ in § 8 follows from the following elegant theorem due to Severi.

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* As a matter of fact it holds for integrals of the second kind of all types, down to Abelian integrals.

In order that $C_1, C_2, \cdots, C_k$ be logarithmic curves of some integral of total differentials, it is necessary and sufficient that they satisfy a relation of equivalence (Severi, *u*, V, p. 209; Lefschetz, *n*, VII, Note I). From this and Picard's result follows our assertion as to ρ.

12. *Genera of Curves. Number of Intersections.* We shall use the following notations (*u*, V; *n*, I):

$[C]$ = genus of the generic curve of $\{C\}$;
$[CD]$ = number of intersections of C and D;
$[C^2]$ = number of intersections of two curves of $\{C\}$.

An extensive symbolic calculus may be developed for these expressions (*n*, I), based on the following two formulas:

$$[(C+D)^2] = [C^2] + 2[CD] + [D^2], \tag{6}$$

$$[(C+D)] = [C] + [D] + [CD] - 1. \tag{7}$$

The proof of (6) is immediate, and that of (7) (Noether, *p*, III; Picard, *q*, II, p. 106) may be carried out very simply as follows. Subdivide a Riemann surface for C into α_2 two-cells with α_1 edges and α_0 vertices. The expression $\alpha_0 - \alpha_1 + \alpha_2$ is independent of the mode of subdivision and when C is irreducible its value is $2 - 2[C]$. Let now C vary and acquire a new double point. With a properly chosen subdivision, it is found that α_1 alone varies, and in fact decreases by 2, so that $\alpha_0 - \alpha_1 + \alpha_2$ is increased by 2. Similarly, if C acquires d double points, the expression increases by $2d$. Let then the generic curves of $\{C\}$, $\{D\}$, $\{C+D\}$ be irreducible, which is the only case of interest, and let α_i', α_i'' be the α_i of D and of the generic curve of $\{C+D\}$. We have at once

$$2[CD] + \alpha_0'' - \alpha_1'' + \alpha_2'' = (\alpha_0 - \alpha_1 + \alpha_2) + (\alpha_0' - \alpha_1' + \alpha_2'),$$

$$2[CD] + 2 - 2[(C+D)] = 2 - 2[C] + 2 - 2[D],$$

whence (7) follows.

The formulas (6) and (7) can be greatly generalized, and in particular lead to a meaning for the characters of any system whatever, even reducible or virtual. More important, however, is the following fact. Let $C_1, C_2, \cdots, C_\rho$ be a base, and let

$$\lambda C = \sum \lambda_i C_i, \qquad \mu D = \sum \mu_i C_i;$$

then (n, I, p. 207)

$$[C] = \prod_{i=1} (1 + C_i)^{\frac{\lambda i}{\lambda}} - \frac{1}{\lambda} \sum_{i=1} \lambda_i, \tag{8}$$

where the product at the right is to be expanded according to the binomial theorem, the terms of degree one and two alone being kept and then replaced by the corresponding character. Similarly (Severi, u, V, p. 223)

$$[CD] = \left[\left(\sum \frac{\lambda_i}{\lambda} \cancel{G_i} \right) \left(\sum \frac{\mu_k}{\mu} C_k \right) \right] = \frac{1}{\lambda\mu} \sum \lambda_i \mu_k [C_i C_k]. \tag{9}$$

C_i

It is clear that (8) and (9) give the genera and the number of intersections of curves in terms of similar data concerning the curves of the base, when their expressions in terms of the base are known. Let, for example, F be a ruled surface, and denote a generator by G. From analysis situs (n, VII, Ch. 3) it follows readily that any $\Gamma_2 \sim \alpha H + \beta G$. In particular any algebraic curve $C \sim \alpha H + \beta G$, and therefore

$$C = \alpha H + \beta G.$$

(See also Severi, u, I, p. 22.) Since $[G_2] = 0$ we have at once $\alpha = [CG]$, number of intersections of C with any G. Then from $[H^2] = m$, $[HG] = 1$, follows $\beta = \mu - m\alpha$, where $\mu = [CH]$, order of C. In terms of α and μ we have $G = \alpha H + (\mu - m\alpha)G$. On remarking that $[G^2] = 0$, $[H] = p$ (here p is the genus of the ruled surface also), (8) gives

$$[C] = 1 + \mu(\alpha - 1) + \alpha(p - 1) - \frac{m\alpha(\alpha - 1)}{2}.$$

Finally, if

$$C' = \alpha' H + (\mu - m\alpha')G,$$

we have from (9)

$$[CC'] = \alpha\mu' + \mu\alpha' - m\alpha\alpha'.$$

Both these formulas are due to Segre (t). Observe that $G = 0$ requires $\alpha = \beta = 0$, which means that H and G satisfy no equivalence; and as they constitute a minimum base, we have $\rho = 2, \sigma = 1$.

13. *Existence Theorem for* $\{C\}$. In terms of the characters $[C]$, $[C^2]$, some noteworthy results concerning the dimension of $|C|$ or $\{C\}$ may be readily expressed. In order to avoid introducing new terms we only give a somewhat special proposition taken from the beautiful theory due mostly to Castelnuovo-Enriques, and Severi (*e*, 706):

Let $s = [C^2] - [C] + 1 - q$. *If* $s \geqq 0$, $\{C\}$ *exists and consists of* ∞^q *linear systems. Should the generic* C *be irreducible,* $|C|$ *is of dimension* $\geqq s$.

14. *Severi's Criterion.* In his first and most important paper on the base, appears this criterion for equivalence:

If A, B *are of the same order and if* $[A^2] = [AB] = [B^2]$ *then* $\lambda A = \lambda B, \lambda \neq 0$.

The proof outlined here constitutes a simplification of Severi's. Let $[B] \geqq [A]$, and set $H_t = H + t(A - B)$. From $[HB] = [HA]$, it follows

$$[H_tH] = [H^2] = m,$$

i.e., the order of H_t is fixed. It is then found, say by (8) and (9), that

$$[H_t^2] - [H_t] + 1 - q = m - p + t[B] - t[A] + 1 - q,$$

which approaches ∞ with t. Hence all systems $\{H_t\}$, with t above a certain limit, exist. As their curves are all of same order, the number of distinct systems among them is finite, so that, for example, $\{H_{t_1}\}$ and $\{H_{t_2}\}$, $t_1 \neq t_2$, coincide. Hence, we have at once

$$(t_1 - t_2)A = (t_1 - t_2)B.$$

From this criterion Severi concludes that in order that $\sum_{i=1}^{k} \lambda_i C_i = 0$, it is necessary and sufficient that the matrix

$$\left\| \begin{matrix} [C_1H], & [C_2H], & \cdots, & [C_kH] \\ [C_1^2], & [C_1C_2], & \cdots, & \cdots \\ \cdot & \cdot & \cdot & \cdot \\ [C_1C_k], & \cdots, & \cdots, & [C_k^2] \end{matrix} \right\| \tag{10}$$

be of rank $< k$.

Suppose for example that $C_1, C_2, \cdots, C_k$ are logarithmic curves of a certain integral of total differentials J with a period α_i with respect to C_i. On any curve D not a C_i there is determined an Abelian integral with the $[DC'_i]$ intersections of C_i and D for logarithmic points, the corresponding periods being α_i. It follows $\sum \alpha_i[DC_i] = 0$. As a matter of fact D may be replaced in this relation by any curve whatever, and therefore the matrix (10) is of rank $< k$. Hence the (C)'s are related by an equivalence. Conversely if they are so related J exists and is readily formed. This is essentially Severi's proof of his theorem stated in No. 11 (u, V).

15. *Determination of* ρ *and* σ. The actual determination of these elements for a given F is a difficult problem. A regular process may be given, but it has little practical value. However, various special methods have yielded the solution in all cases of interest. These cases fall mainly into two classes which we examine in turn.*

The first class consists of surfaces of sufficiently general nature contained in known algebraic varieties. The simple varieties (linear spaces or their loci, Abelian varieties) yield the most significant results. As early as 1882, Noether stated (p, II) the following important theorem: An arbitrary non-singular surface of order $m > 3$ situated in an S_3 (S_r = r-space) contains only curves that are complete intersections with other surfaces of S_3. It follows that for such a surface a plane section constitutes a minimum base, and therefore $\rho = \sigma = 1$. An incomplete proof of a geometric nature, of a similar theorem for surfaces that are complete intersections of $r - 2$ varieties in an S_r was given by Fano (j, II).†

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* In e (p. 730) will be found references pertaining to surfaces not discussed here.

† He makes an assumption leading to the impossibility for a double integral of the first kind to be without periods. The proof in n, IV, p. 358, is correct but for this exception noticed by Fano: the quartic surface intersection of two quadrics in S_4. The proof (loc. cit.) fails when the integer denoted there by n is negative, and a very simple discussion shows that when r exceeds three, this occurs only in the case just mentioned.

As beyond the scope of this report but noteworthy here must be mentioned extensions to algebraic varieties (Klein, m; Fano, j, I; Severi, u, VIII; Lefschetz, n, IV, p. 359).

A very general theorem from which the preceding may be derived was given by the writer by means of analysis situs (*n*, IV, p. 355). The following important special case will suffice as an indication. In a three-dimensional variety V_3 let $|F|$ be a linear system of surfaces, ∞^3 at least, of sufficiently general type (more or less analogous to the system of hyperplane sections). If the number of linearly independent double integrals of the second kind of a generic F exceeds that of V_3 (see *n*, IV), a base for the surfaces of V_3 intersects F into a similar one for its curves. V_3 and F have then equal numbers ρ, σ. (These numbers and the bases are defined for V_3 as for a surface.) Thus, to prove Noether's theorem, it is sufficient to show that F possesses double integrals of the first kind with periods not all zero, which can be done in this case (*n*, IV, p. 358; *n*, VII, Ch. 5). These will constitute integrals of the second kind not of type (5) (Picard, *q*, II, p. 365).

As S_3 possesses no double integrals of the second kind that are linearly independent, the theorem becomes applicable. A plane constitutes a minimum base for S_3 and its trace on F, that is a plane section, will be one for F. The surface being regular, Noether's theorem follows readily.

The second class of surfaces for which the determination of the bases has been carried out is composed of hyperelliptic surfaces and surfaces which are the image of pairs of points of two algebraic curves.

In a series of papers dating from 1893, G. Humbert made a searching investigation of hyperelliptic surfaces (*l*). Applying a result due to Appell, he showed in particular that the parametric equation of any curve traced on a hyperelliptic surface F is characterized by the vanishing of an intermediary function attached to the period matrix (entire function such that the addition of periods has merely the effect of multiplying it by a linear exponential) (*l*, first paper). From this and through an elegant analysis, Bagnera and de Franchis obtained the value of ρ and the bases even for hyperelliptic surfaces of very special type (*d*).

Surfaces images of point-pairs of two algebraic curves have been investigated by various authors (*e*, p. 730), the most important contribution being Severi's (*u*, I). In substance he shows that to every correspondence between the two curves corresponds a curve on F, and vice versa. Let λ be the number of singular correspondences between the two curves in the sense of Hurwitz. Severi's result gives readily $\rho = \lambda + 2$, $\sigma = 1$. However, he did not state this explicitly (loc. cit.) for his paper antedates by three years his first one on the base.

The surfaces considered in this article furnish ideal applications for the theorem (*b*) of § 8 (*n*, VII, Ch. 4). Results already known are obtained with great ease and elegance. Moreover the same method has been the basis for the extension to Abelian varieties (*n*, IV).

16. *Conclusion.* An outstanding question is the determination of ρ and the bases when only real curves are taken into consideration. So far as we know it has been solved only for rational surfaces (Comessati, *g*) and hyperelliptic surfaces (Lefschetz, *n*, V).

Algebraic varieties which we have excluded from this report are still somewhat *terra incognita,* although some important general theorems are known (*n*, IV). Perhaps we must await further information of a purely geometric nature before much progress can be expected.

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THE UNIVERSITY OF KANSAS

NOTE ON THE CONVERGENCE OF WEIGHTED TRIGONOMETRIC SERIES*

BY DUNHAM JACKSON

1. *Introduction.* Let $f(x)$ be a function continuous for all values of x, and of period 2π. Let $T_n(x)$ be a trigonometric sum of the nth order.† If $T_n(x)$ is determined, among all such sums, by the condition that the value of the integral

$$\int_0^{2\pi}[f(x) - T_n(x)]^2 dx$$

shall be a minimum, it becomes the partial sum of the Fourier series for $f(x)$. The problem can be generalized by taking, as the quantity to be reduced to a minimum, the integral

$$\int_0^{2\pi}\rho(x)[f(x) - T_n(x)]^2 dx, \tag{1}$$

where $\rho(x)$, indicating the *weight* to be attached to different values of the argument, is a function of x, likewise of period 2π, and positive for all values of x. There is a considerable body of literature bearing more or less directly on the generalized problem. This literature owes its inspiration largely to the researches of Tchebychef;‡ particular mention should also be made of a classical memoir by Gram.§

The purpose of the following paragraphs is to discuss the convergence of $T_n(x)$ toward the value $f(x)$, as n becomes infinite. The method is one which I have used recently in connection with the corresponding problem in which the weight is constantly equal to unity, and the square of the error is replaced by a power with a different exponent. The question of convergence is treated by Gram, in the paper cited, but scarcely in a manner to meet the requirements of modern analysis.* More recently it has come within the range of a number of investigations, including a series of papers by Stekloff, in the BULLETIN DE L'ACADÉMIE DES SCIENCES, PETROGRAD, and elsewhere, with which I am only very imperfectly acquainted; a paper by J. Chokhate,† which I have seen in manuscript; and a series of papers by Szegö.‡ Up to the present time, I have not seen any treatment covering precisely the results that are presented below. If it should appear nevertheless that such a treatment exists, the novelty of this paper would consist in the method employed, and in the applicability of the method to the case in which the exponent 2 in (1) is replaced by an arbitrary m, as suggested in the concluding paragraph.

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* Presented to the Society, December 30, 1920.

† The words "of the nth order" will be understood throughout to mean "of the nth order at most."

‡ Cf., e.g., H. Burkhardt, *Entwicklungen nach oscillirenden Functionen und Integration der Differentialgleichungen der mathematischen Physik*, JAHRESBERICHT DER VEREINIGUNG, vol. 10, Heft 2 (1908), pp. 823 ff.

§ J. P. Gram, *Ueber die Entwickelung reeller Functionen in Reihen mittelst der Methode der kleinsten Quadrate*, JOURNAL FÜR MATHEMATIK, vol. 94 (1883), pp. 41–73.

2. *The Convergence Theorem.* The conclusion to be established is as follows:§

Let $\omega(\delta)$ be the maximum of $|f(x') - f(x'')|$ for $|x' - x''| \leqq \delta$. Let $\rho(x)$ be continuous and positive for all values of x; or, if not continuous, let it be measurable, and always included between two fixed positive bounds.|| Then we may state the theorem:

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* Cf. Burkhardt, loc. cit., pp. 848–854.

† See also J. Chokhate, *Sur quelques propriétés des polynomes de Tchébicheff*, COMPTES RENDUS, vol. 166 (1918), pp. 28–31.

‡ G. Szegö, *Über die Entwickelung einer analytischen Funktion nach den Polynomen eines Orthogonalsystems*, MATHEMATISCHE ANNALEN, vol. 82 (1921), pp. 188–212; *Über die Entwicklung einer willkürlichen Funktion nach den Polynomen eines Orthogonalsystems*, MATHEMATISCHE ZEITSCHRIFT, vol. 12 (1922), pp. 61–94; *Über den asymptotischen Ausdruck von Polynomen, die durch eine Orthogonalitätseigenschaft definiert sind*, MATHEMATISCHE ANNALEN, vol. 86 (1922), pp. 114–139; and other papers referred to in footnotes attached to the above.

§ The proof of the existence of a unique solution for the minimum problem is based so directly on similar proofs already given that it will not be taken up in detail here; cf. D. Jackson, *On functions of closest approximation*, TRANSACTIONS OF THIS SOCIETY, vol. 22 (1921), pp. 117–128.

|| It would of course make no difference if this condition were violated at points of a set of measure zero, since the value of the integral (1), and consequently the determination of $T_n(x)$, would not be affected.

The sum $T_n(x)$ *will converge uniformly to the value* $f(x)$ *for* $n \to \infty$ *provided that*[*]

$$\lim_{\delta \to 0} \omega(\delta)/\sqrt{\delta} = 0.$$

As already stated, the proof is similar to one given recently in another connection.[†] In the first place, if $f(x)$ and $\varphi(x)$ are two functions whose difference is a trigonometric sum $t_n(x)$ of order n:

$$f(x) = \varphi(x) + t_n(x),$$

and if $T_n(x)$ and $\tau_n(x)$ are two sums, likewise of order n, such that

$$T_n(x) = \tau_n(x) + t_n(x),$$

the value of the integral (1) formed with $f(x)$ and $T_n(x)$ is the same as the value of the corresponding integral formed with $\varphi(x)$ and $\tau_n(x)$, and both integrals will reach their minimum values simultaneously. That is, if $T_n(x)$ and $\tau_n(x)$ represent the best approximating functions for $f(x)$ and $\varphi(x)$, respectively, as judged by the value of the integral (1), the errors $f(x) - T_n(x)$ and $\varphi(x) - \tau_n(x)$ will be identical.

By a general theorem on the approximate representation of continuous functions,[‡] there will exist sums $t_n(x)$, of all orders $n > 0$, such that the difference between $f(x)$ and $t_n(x)$ never exceeds a constant multiple of $\omega(2\pi/n)$. In formulas, let

$$\varphi_n(x) = f(x) - t_n(x),$$

and let ϵ_n be the maximum of $|\varphi_n(x)|$; then

$$\epsilon_n \leqq c\omega(2\pi/n),$$

where c is independent of n. In particular, if $\omega(\delta)$ satisfies the hypothesis of the theorem,

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[*] There is no reason to suppose that the particular infinitesimal $\sqrt{\delta}$ has any essential significance for the problem; its occurrence is in all probability due merely to the limitations of the method.

[†] D. Jackson, *On the convergence of certain trigonometric and polynomial approximations*, TRANSACTIONS OF THIS SOCIETY, vol. 22 (1921), pp. 158–166.

[‡] Cf., e.g., D. Jackson, *On the approximate representation of an indefinite integral and the degree of convergence of related Fourier's series*, TRANSACTIONS OF THIS SOCIETY, vol. 14 (1913), pp. 343–364; p. 350.

$$\lim_{n\to\infty} \epsilon \sqrt{n} = 0. \tag{2}$$

Let $\tau_n(x)$ be the trigonometric sum of order n which gives the best approximation to $\varphi_n(x)$, as determined by the integral corresponding to (1); let

$$\gamma_n = \int_0^{2\pi} \rho(x)[\varphi_n(x) - \tau_n(x)]^2 dx;$$

and let $\mu_n = |\tau_n(x_0)|$ be the maximum of $|\tau_n(x)|$. Let it be assumed that

$$0 < v \leqq \rho(x) \leqq V,$$

the numbers v and V being constants; and let it be assumed temporarily that $\mu_n \geqq 4\epsilon_n$.

By Bernstein's theorem,* since

$$|\tau_n(x)| \leqq \mu_n,$$

it follows that

$$|\tau_n'(x)| \leqq n\mu_n$$

for all values of x. In particular, for values of x in the interval

$$|x - x_0| \leqq \frac{1}{2n},$$

it can be inferred that

$$|\tau_n(x) - \tau_n(x_0)| \leqq \frac{\mu_n}{2},$$

and

$$|\tau_n(x)| \geqq \frac{\mu_n}{2}.$$

Since

$$|\varphi_n(x)| \leqq \epsilon_n \leqq \mu_n/4,$$

it follows further that

$$|\varphi_n(x) - \tau_n(x)| \geqq \frac{\mu_n}{4}$$

throughout the interval specified, and, as the length of the interval is $1/n$, and $\rho(x) \geqq v$,

$$\gamma_n \geqq \frac{v}{n}\left(\frac{\mu_n}{4}\right)^2.$$

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* See, e.g., de la Vallée Poussin, *Leçons sur l'Approximation des Fonctions d'une Variable Réelle*, Paris, 1919, pp. 39–42.

On the other hand, by the minimum property of $\tau_n(x)$, the value of γ_n is less than that which would be obtained if $\tau_n(x)$ were replaced by any other trigonometric sum of order n; in particular, by comparison with the integral which is obtained if 0 is substituted for $\tau_n(x)$,

$$\gamma_n \leqq 2\pi V \epsilon_n^2.$$

Hence

$$\frac{v}{n}\left(\frac{\mu_n}{4}\right)^2 \leqq 2\pi V \epsilon_n^2,$$

$$\mu_n \leqq 4\sqrt{\frac{2\pi V}{v}}\ \epsilon_n \sqrt{n}.$$

This relation, derived on the hypothesis that $\mu_n \geqq 4\epsilon_n$, clearly holds in the contrary case also, since $V \geqq v$ and $n \geqq 1$.

In any case, then, since $|\varphi_n| \leqq \epsilon_n$ and $|\tau_n| \leqq \mu_n$,

$$|\varphi_n(x) - \tau_n(x)| \leqq \epsilon_n + 4\sqrt{\frac{2\pi V}{v}}\ \epsilon_n \sqrt{n} \leqq k\epsilon_n \sqrt{n},$$

where k is independent of n. But it has been pointed out already that $\varphi_n(x) - \tau_n(x)$ is the same as $f(x) - T_n(x)$, where $T_n(x)$ is the sum giving the best approximation to $f(x)$, as determined by the integral (1); hence

$$|f(x) - T_n(x)| \leqq k\epsilon_n \sqrt{n}.$$

This relation, combined with (2), establishes the truth of the theorem.

With the same method of treatment, the problem can be varied by using a general power of the absolute value of the error, instead of the square, together with a weight-function $\rho(x)$; and the method is applicable also to problems of polynomial approximation. For treatment in detail, however, the case discussed above may be regarded as sufficiently illustrative.

THE UNIVERSITY OF MINNESOTA

PROOF OF A FORMULA FOR AN AREA*

BY H. E. BRAY

1. *Introduction.* The formula

$$A = \int\int_R \frac{\partial(x, y)}{\partial(u, v)} dudv,$$

representing the area of the image of a (u, v) rectangle R in the (x, y) plane, can be proved to hold in the case where the transformation $x = x(u, v)$, $y = y(u, v)$ is of a very general kind.† The purpose of this paper is to extend and prove the formula by means of an approximating function of a very simple kind. The main properties of this function are given in § 3 and used in the proof of the formula, § 4.

2. *Definitions.* The approximating function for the summable function $f(x, y)$ is given by the formula

$$f^{(\mu)}(x, y) = \frac{1}{\sigma_\mu}\int\int_{\sigma_\mu} f(x + \xi, y + \eta) d\xi d\eta$$

where σ_μ represents the square region $[-\mu \leqq \xi \leqq \mu, -\mu \leqq \eta \leqq \mu]$, also the area of that square.‡ For convenience $f(x, y)$ is regarded as summable (Lebesgue) in the fundamental region S $[0 \leqq x \leqq 1, 0 \leqq y \leqq 1]$ and the properties of $f^{(\mu)}$ are considered with reference to a rectangle R $[a \leqq x \leqq b, c \leqq y \leqq d]$ inside S, μ being less than a, c, $1 - b$, $1 - d$.

The formula to be proved involves *generalized derivatives* and *potential functions.* These are defined as follows:

DEFINITION (i). If α is a given direction and α' the direction 90° in advance of α, the quantity

$$D_\alpha f(x, y) = \lim_{\sigma \to 0} \frac{1}{\sigma} \int_s f(x, y) d\alpha',$$

if it exists, is called the *generalized derivative of f in the direction α.* It is understood that the integral is taken in the positive

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[*] Presented to the Society, December 27, 1922.

[†] W. H. Young, PROCEEDINGS OF THE LONDON SOCIETY, (2), vol. 18, p. 339.

[‡] It is convenient sometimes to use as σ_μ the interior of the circle $\xi^2 + \eta^2 = \mu^2$. The properties of $f^{(\mu)}$ are essentially the same in this case.

sense around the contour s, of a square σ, having its center at (x, y) and its sides parallel to the coordinate axes.§

DEFINITION (ii). Let $\varphi(x, y)$ be a vector point function whose component φ_α in every fixed direction, α, is summable superficially, and $u(x, y)$ a scalar point function, summable superficially, and such that $\int_s u(x, y)d\alpha$ may be defined for every given direction α. Then if φ and u are so related that $\int\int_\sigma \varphi_\alpha d\sigma = \int_s u d\alpha'$ for every rectangle σ * and for every fixed direction α, u is said to be a *potential function for the vector* φ.

3. *Properties of the Function* $f^{(\mu)}$.†

(1) If $f(x, y)$ is summable superficially $f^\mu(x, y)$ is a continuous function of (x, y), absolutely continuous in x for every y and in y for every x. At nearly every point of any line $y =$ const. $\partial f^{(\mu)}/\partial x$ exists:

$$\frac{\partial}{\partial x} f^{(\mu)} = \frac{1}{\sigma_\mu} \int_{s_\mu} f(x + \xi, y + \eta) d\eta,$$

the integral being evaluated about the contour s_μ of σ_μ.‡

(2) If $f(x, y)$ is absolutely continuous in x for every y and if $\partial f/\partial x$ is summable superficially, then, if y is given, for nearly every value of x,

$$\frac{\partial}{\partial x} f^{(\mu)} = \left(\frac{\partial f}{\partial x}\right)^{(\mu)}.$$

(3) If $f(x, y)$ is a potential function for its generalized derivatives, then, if y is given, for nearly every x

$$\frac{\partial}{\partial x} f^{(\mu)} = (D_x f)^\mu.$$

(4) If $f(x, y)$, besides satisfying the conditions of (3), is also

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§ G. C. Evans, *Fundamental problems of potential theory*, RICE INSTITUTE PAMPHLETS, vol. 7, No. 4 (Oct., 1920). In the original definition Professor Evans considers a general class Γ of curves. A summary of this paper is given in PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 7 (1921).

* Here, again, the definition as given by Professor Evans refers to the general class Γ of curves.

† The proofs are simple and are omitted, in most cases, for brevity.

‡ $f^{(\mu)}$ can be generalized when higher derivatives are required; e.g.,

$$f^\mu(x, y) = \frac{1}{\sigma_\mu{}^2} \int\int_{\sigma_\mu} d\xi' d\eta' \int\int_{\sigma_\mu} f(x + \xi + \xi', y + \eta + \eta') d\xi d\eta$$

possesses second derivatives nearly everywhere.

continuous in (x, y), then

$$\frac{\partial}{\partial x} f^{(\mu)} = (D_x f)^{(\mu)},$$

at every point (x, y).

(5) If $f(x, y)$ is a potential function for its generalized derivatives, then $D_x(f^{(\mu)})$ exists and is equal to $(D_x f)^{(\mu)}$.

(6) If $f(x, y)$ is summable superficially, the function $f(x+\xi, y+\eta)$ is measurable and summable in the three-dimensional region $[a \leqq s \leqq b, -\mu \leqq \xi \leqq \mu, -\mu \leqq \eta \leqq \mu]$, where s is the arc of a rectifiable curve upon which the point (x, y) lies, and

$$\int_{s_1}^{s_2} f^{(\mu)}(x, y) ds = \frac{1}{\sigma_\mu} \int_{s_1}^{s_2} ds \int\int_{\sigma_\mu} f(x+\xi, y+\eta) d\xi d\eta$$
$$= \frac{1}{\sigma_\mu} \int\int_{\sigma_\mu} d\xi d\eta \int_{s_1}^{s_2} f(x+\xi, y+\eta) ds.$$

(7) Similarly $f(x+\xi, y+\eta)$, regarded as a function of four variables (x, y, ξ, η), is summable in the region

$$[a \leqq x \leqq b, \quad c \leqq y \leqq d, \quad -\mu \leqq \xi \leqq \mu, \quad -\mu \leqq \eta \leqq \mu].$$

Consequently if σ is the rectangle $[a \leqq x \leqq b, c \leqq y \leqq d]$

$$\int\int_{\sigma} f^{(\mu)}(x, y) dx dy = \frac{1}{\sigma_\mu} \int\int_{\sigma} dx dy \int\int_{\sigma_\mu} f(x+\xi, y+\eta) d\xi d\eta$$
$$= \frac{1}{\sigma_\mu} \int\int_{\sigma_\mu} d\xi d\eta \int\int_{\sigma} f(x+\xi, y+\eta) dx dy.$$

(8) Since $\int\int_\sigma f(x+\xi, y+\eta) dx dy$ is a continuous function of (ξ, η),

$$\lim_{\mu \to 0} \int\int_{\sigma} f^{(\mu)}(x, y) dx dy$$
$$= \lim_{\sigma_\mu \to 0} \frac{1}{\sigma_\mu} \int\int_{\sigma_\mu} d\xi d\eta \int\int_{\sigma} f(x+\xi, y+\eta) dx dy$$
$$= \int\int_{\sigma} f(x, y) dx dy.$$

(9) If $f(x, y)$ and $g(x, y)$ are two functions, summable with their squares, then

$$\lim_{\mu \to 0} \int\int_{\sigma} f^{(\mu)} g^{(\mu)} dx dy = \int\int_{\sigma} f g dx dy.$$

In fact since f^2 and g^2 are summable, it can be shown by

Schwarz's inequality that the absolute continuity of the integral $\int\int |f^{(\mu)}g^{(\mu)}|dxdy$ is uniform* for all μ, and this is sufficient to prove the property.

(10) If $f(x, y)$ is continuous in (x, y), $f^{(\mu)}$ approaches f uniformly as μ approaches 0. If f is merely summable, $f^{(\mu)}$ approaches f at nearly every point (x, y).

(11) If $f(x, y)$ is a function of limited variation in x, uniformly for every y, then $f^{(\mu)}$ is a function of limited variation in x, uniformly for all values of y and μ. In fact, if

$$\sum_i |f(x_{i+1}, y) - f(x_i, y)| \leqq T_x,$$

a constant, then

$$\sum_i |f^{(\mu)}(x_{i+1}, y) - f^{(\mu)}(x_i, y)|$$

$$\leqq \frac{1}{\sigma_\mu}\int\int_{\sigma_\mu} \sum_i |f(x_{i+1}+\xi, y+\eta) - f(x_i+\xi, y+\eta)|d\xi d\eta \leqq T_x.\dagger$$

4. *Proof of the Formula.* It has been shown by W. H. Young‡ that, given the continuous transformation $x = x(u, v)$, $y = y(u, v)$, the area of the image of the rectangle R $[a \leqq u \leqq b$, $c \leqq v \leqq d]$ is given by the formula

$$A = \int\int_R \frac{\partial(x, y)}{\partial(u, v)}\, dudv.$$

He assumes the absolute continuity of x and y as functions of u alone and of v alone, the absolute continuity with regard to u being uniform for a dense set of values of v in the interval $c \leqq v \leqq d$. In the theorem which follows absolute continuity of x and y is not assumed; instead it is assumed that x and y are potential functions for their generalized derivatives.

THEOREM. *If $x(u, v)$, $y(u, v)$ are continuous functions of (u, v), if each is a potential function for its generalized derivatives, the latter being summable with their squares, and if y (or x) is a function of uniformly limited variation in u for every v, and in v for every u, then the area of the image of the rectangle*

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* C. de la Vallée-Poussin, TRANSACTIONS OF THIS SOCIETY, vol. 16 (1915), pp. 445 et seq.

† It is sufficient to assume that the total variation with regard to x is less than some function of y, $c(y)$, summable in y and such that

$$\int_y^{y'} c(y)dy < M(y' - y).$$

‡ W. H. Young, loc. cit.

$R\,[a \leqq u \leqq b,\ c \leqq v \leqq d]$ is given by the formula

$$A = \int\!\!\int_R (D_u x D_v y - D_v x D_u y)\,du\,dv.$$

It is shown by Young* that the area A is equal to

$$\tfrac{1}{2}\int_{C_R} x\,dy - y\,dx,$$

where the integrals are Stieltjes integrals evaluated about the contour C_R of R. By a known property of the Stieltjes integral,† namely,

$$\int_a^b x\,dy = xy\Big]_a^b - \int_a^b y\,dx,$$

it follows, under our hypothesis, that

$$A = \int_{C_R} x\,dy.$$

Instead of this expression we consider the approximation

$$A_\mu = \int_{C_R} x^{(\mu)}dy^{(\mu)} = \int_{C_R} x^{(\mu)}\frac{\partial}{\partial u}y^{(\mu)}du + x^{(\mu)}\frac{\partial}{\partial v}y^{(\mu)}dv.$$

The change to the last form is justified by the fact that $x^{(\mu)}$, $y^{(\mu)}$ are continuous with their first derivatives. By property (4) this can be written in the form

$$\int_{C_R} x^{(\mu)}(D_u y)^{(\mu)}du + x^{(\mu)}(D_v y)^{(\mu)}dv,$$

and since the integrands are absolutely continuous

$$A_\mu = \int_a^b du\int_c^d -\frac{\partial}{\partial v}[x^{(\mu)}(D_u y)^{(\mu)}]dv + \int_c^d dv\int_a^b \frac{\partial}{\partial u}[x^{(\mu)}(D_v y)^{(\mu)}]du.$$

The quantity $\partial(D_v y)^{(\mu)}/\partial u$ is summable superficially, being equal to the contour integral of $D_v y$ about a square (properties (1) and (6)). Moreover for every rectangle $r\,[a' \leqq u \leqq b',\ c' \leqq v \leqq d']$,

$$\int\!\!\int_r \left[\frac{\partial}{\partial u}(D_v y)^{(\mu)}\right]du\,dv = \int\!\!\int_r \left[\frac{\partial}{\partial v}(D_u y)^{(\mu)}\right]du\,dv,$$

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* W. H. Young, loc. cit.

† H. E. Bray, Annals of Mathematics, (2), vol. 20 (1918-19), p. 185.

Since

$$\int\!\!\int_r \left[\frac{\partial}{\partial u}(D_v y)^{(\mu)}\right] dudv = \left[\int_{c'}^{d'} (D_v y)^{\mu} dv\right]_{u=a'}^{u=b'}$$

$$= \left[\int_{c'}^{d'} \frac{\partial}{\partial v} y^{(\mu)} dv\right]_{u=a'}^{u=b'}$$

$$= y^{(\mu)}(b', d') - y^{(\mu)}(b', c') - y^{(\mu)}(a', d') + y^{(\mu)}(a', c').$$

Consequently, x being continuous, for every rectangle r,

$$\int\!\!\int_r x^{(\mu)} \frac{\partial}{\partial u}(D_v y)^{(\mu)} dudv = \int\!\!\int_r x^{(\mu)} \frac{\partial}{\partial v}(D_u y)^{(\mu)} dudv.$$

We can now express the quantity A_μ as a double integral and on cancelling the terms just mentioned we obtain

$$A_\mu = \int\!\!\int_R \left[\frac{\partial}{\partial u} x^{(\mu)}(D_v y)^{(\mu)} - \frac{\partial}{\partial v} x^{(\mu)}(D_u y)^{(\mu)}\right] dudv,$$

and since

$$\frac{\partial}{\partial u} x^{(\mu)} = (D_u x)^{(\mu)}, \qquad \frac{\partial}{\partial v} x^{(\mu)} = (D_v x)^{(\mu)},$$

by property (4),

$$A_\mu = \int\!\!\int_R [(D_u x)^{(\mu)}(D_v y)^{(\mu)} - (D_v x)^{(\mu)}(D_u y)^{(\mu)}] dudv.$$

Now let μ approach zero. Since the generalized derivatives are, by hypothesis, summable with their squares, by (9),

$$\lim_{\mu \to 0} A_\mu = \int\!\!\int_R [D_u x D_v y - D_v x D_u y] dudv.$$

To show that $\lim_{\mu \to 0} A_\mu = \int_{C_R} xdy$, consider the following quantity, in which the integrals are taken in regard to u for a fixed value of v:

$$\left| \int_{u=a}^{u=b} x^{(\mu)} dy^{(\mu)} - \int_{u=a}^{u=b} xdy \right|$$

$$\leqq \left| \int x^{(\mu)} dy^{(\mu)} - \int x dy^{(\mu)} \right| + \left| \int x dy^{(\mu)} - \int xdy \right|$$

$$= \left| \int (x^{\mu} - x) dy^{(\mu)} \right| + \left| \int x dy^{(\mu)} - \int xdy \right| \cdot$$

The first term is not greater than $\max|x^{(\mu)} - x|$ times the total variation of $y^{(\mu)}$; and since the latter is uniformly limited for every μ and $x^{(\mu)}$ approaches x uniformly, this term approaches zero with μ. The second term approaches zero because $y^{(\mu)}$ approaches y and is of uniformly limited variation in u for every μ,* by (11). The quantity $\int_{u=a}^{u=b} x^{(\mu)}dy^{(\mu)}$ is typical of those which constitute A_μ, consequently

$$\lim_{\mu\to 0} A_\mu = \int_{C_R} xdy = A.$$

We have thus proved that

$$A = \int\int [D_u x D_v y - D_v x D_u y]dudv.$$

It has been shown by Evans † that if a continuous function is a potential function for its generalized derivatives, then its ordinary derivatives exist nearly everywhere and are equal to the corresponding generalized derivatives nearly everywhere. The formula which we have proved remains true, consequently, if ordinary derivatives are substituted for the generalized derivatives. On the other hand, if $\partial x/\partial u$, $\partial x/\partial v$ are given, a vector φ is defined whose components in the u and v directions are $\partial x/\partial u$, $\partial x/\partial v$, respectively, and

$$\varphi_\alpha = \frac{\partial x}{\partial u}\cos(u, \alpha) + \frac{\partial x}{\partial v}\sin(u, \alpha).$$

If $\partial x/\partial u$, $\partial x/\partial v$ are summable the same is true for any component φ_α and if

$$\int\int_\sigma \frac{\partial x}{\partial u}dudv = \int_{C_\sigma} xdv, \qquad \int\int_\sigma \frac{\partial x}{\partial v}dudv = -\int_{C_\sigma} xdu,$$

x will be a potential function for the vector φ which is called its gradient vector.‡ Hence we have the following theorem.

THEOREM. *If $x(u, v)$, $y(u, v)$ are continuous functions of (u, v), if each is a potential function for its gradient vector, if y (or x) is of uniformly limited variation in u for every v and in v for every u, and if $\partial x/\partial u$, $\partial x/\partial v$, $\partial y/\partial u$, $\partial y/\partial v$ are summable with their squares, then*

$$A = \int\int_R \frac{\partial(x, y)}{\partial(u, v)}dudv.$$

RICE INSTITUTE

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[*] H. E. Bray, loc. cit.

[†] G. C. Evans, RICE INSTITUTE PAMPHLETS, vol. 7, No. 4, pp. 97–99.

[‡] G. C. Evans, loc. cit.

MATHEMATICS FOR THE LAYMAN

Mathematical Philosophy. A Study of Fate and Freedom. By Cassius J. Keyser. New York, E. P. Dutton and Co., 1922. 14 + 466 pp.

An author who attempts to give a popular exposition of any technical doctrine, be it in science, in art, or in philosophy, who (to use Professor Keyser's own words) aims to make a "contribution to the democratization of science and scientific criticism," faces a peculiarly difficult dilemma. He must beguile the layman's interest without being superficial; he must achieve adequate depth without being dull. A work of popularization, moreover, to fulfill its highest mission must be not merely descriptive or informative, but also interpretative. No one would probably deny the crying need of our age for popularizing books in all fields of technical inquiry satisfying these demands. And yet how meager the supply!

Professor Keyser has succeeded in meeting these conflicting demands in a remarkable way. His lectures were prepared primarily for students of philosophy; his book is addressed, however, to "educated laymen" in general. The appeal of its interest is very wide. It aims to discuss "the nature of mathematics, its significance in thought, and its bearings on human life." A large program, carried through with notable success! This does not mean, of course, that the last word has been said, that there is not room for difference of opinion, that many important fields of inquiry have not been left untouched. It does mean, however, that Professor Keyser has made a very significant contribution to the solution of the problem he set himself and that he has set up a standard of excellence which books of similar purpose and scope will find it difficult to meet.

The introductory lecture contains a discussion of the fundamental aims of education which is one of the finest things in the book. Many of our influential educators would confer a boon upon their country were they to repair to a mountain top, far away from the tangled undergrowth of short-sighted policy, and there read and ponder this discussion and let their souls commune with the educational ideal there presented. Such an educator might return to the uncharted wilderness of practical problems with a new vision and a more reliable compass.

The next eight lectures deal with various aspects of postulate systems. The point of view is that of Russell and Whitehead and much emphasis is laid on the alleged identity of pure mathematics and symbolic logic. Personally I would question the desirability of arbitrarily restricting so well established a term as "pure mathematics" to the meaning implied in such identification. Why not call the class of doctrinal functions "abstract mathematics" or "formal mathematics," or, if one really believes in the alleged identity, "formal logic"? Quite aside from this verbal problem, however, it seems to me that there is a real distinction between the problem of formal logic and the investigation of the various doctrinal functions—

a distinction at least of motive which I think is vital and should not be obscured. The motive of the logician is logical consistency or the validity of mental processes as such, the motive of the mathematician is the conclusions reached by such processes, their validity being by him assumed. Professor Keyser himself is not altogether consistent in his treatment of this subject. We may well argue that pure mathematics and formal logic are "organically related as are the roots of a tree and its branches," without thereby asserting that the roots and the branches are "the same thing." Furthermore, the definition of mathematics as a class of doctrinal functions—especially in the older form of Russell as "The class of all propositions of the form P implies Q"—leaves out, it seems to me, an essential characteristic of mathematics, viz., its structure. Mathematics is not a mere class—it is at least an ordered class of some sort.

A first example to illustrate the possibility of difference of opinion. Some others may appear later in this review. Let me say immediately that these opportunities for questioning constitute one of the most valuable features of the book. It is stimulating and thought provoking to a high degree.

I have used the phrase "doctrinal function." It is a very happy one and constitutes, I venture to say, a permanent addition to the vocabulary of our science. In proposing it Professor Keyser has done more than merely give an appropriate label. The concept which it describes was doubtless latent in the minds of many before its introduction. But by giving it a name he has precipitated the concept in precise form—and the concept is a valuable one.

Throughout these first nine lectures he develops also the thesis implied by his subtitle "A Study of Fate and Freedom." He calls attention at the very beginning to the eternal quality of ideas and their interrelations, to the immutable laws of thought to which the intellect is subject. "The world of ideas is the empire of Fate" (p. 5). After having developed the concept of a postulate system he returns to his thesis as follows (p. 136): "When once the principles, or postulates, are chosen, the die is cast—all else follows with a necessity, a compulsion, an inevitability that are absolute —we are at once subject to a destiny of consequences which no man nor any hero nor Zeus nor Yahweh nor any god can halt, annul or circumvent. Mathematics is in a word the study of Fate. Let me hasten to say that the Fate is not physical, it is spiritual. . . . The Fate is logical Fate. Is it a tyrant? And the intellect, then, a slave? . . . Where then is the intellect's freedom? What do you love? Poetry? Painting? . . . Music? The muses are *their* fates. If you love them you are free. Logic is the muse of *thought*. When I violate it, I am erratic; if I hate it, I am licentious or dissolute; if I love it, I am free—the highest blessing the austerest muse can give."

Here and throughout the book the author makes out a strong case for Fate—but it seems to me he leaves the case for Freedom unnecessarily weak. So much could have been said—and said beautifully, eloquently, by this particular author. I am sorry he did not say it. Is it true that the die is cast when the postulates have been chosen? What of the wealth of definitions that are possible under a given set of postulates? What of the infinite variety of combinations of the primitive elements which the creative imagination can construct? Royce has called it the "eternal fairyland of mathematical construction." The author would perhaps challenge this point of view—would claim that the imagination does not *create* these combinations, that they are present from the start, and that the mind merely *discovers* them. Some of the author's remarks on the early pages of the book seem to imply this attitude. It seems to me, however, that such an attitude, while perhaps philosophically tenable, is logically unnecessary and esthetically unsatisfying. Did Beethoven discover his Fifth Symphony, or did he create it? Did Georg Cantor discover his Mengenlehre or did he create it? Was the group concept, were the manifold, beautiful developments in the recent theory of functions of real variables discovered or created? No dogmatic answer is perhaps possible —in that the answer depends on the philosophical temperament of the one who replies. But it seems to me at any rate far more satisfying, and equally true, to regard such conceptions as the creations of an artistic imagination. The recent work on the foundations seems to me to have contributed not a little to a better understanding of mathematics as a fine art, like music or painting. That Professor Keyser is fully sympathetic toward such a conception no one can doubt—it is, therefore, a bit puzzling that he does not make more of it in this book. The volume is in itself a fine example of the play of imagination, but it has little to say of imagination as such and the rôle it plays in mathematics.

After his treatment of postulate systems follow lectures on Transformation, Invariance, Group Concept, Limits, Infinity, Hyperspaces, Noneuclidean Geometries, Psychology, Korzybski's Concept of Man, and Science and Engineering. These lectures are so good that they deserve a critical and discriminating review. However, space forbids; and Professor Keyser has assured me that he would prefer a review of general character, as more in keeping with the purpose of the book. Furthermore, an adequate appreciation and criticism must be a composite from many sources. Let philosophers quarrel with the author, if they must, concerning his philosophical tenets; let scientists object, if they will, to the proposition that "mathematics is the prototype which every branch of science approximates in proportion as its basal assumptions and concepts become clearly defined" (but let them not venture into the argument until they have an adequate conception of the broad sense in which the term mathematics is used); let literary critics challenge, if they be so disposed, the author's analysis of the function and art of criticism; and let sociologists and engineers evaluate the author's conception of their proper function in accordance with Korzybski's idea of the nature of our human kind.

Enough has already been said to indicate my conviction that Professor Keyser has made a valuable contribution to the literature of popularization in the highest and best sense of this much abused term. His book accurately presents the more fundamental conceptions which form the basis of modern mathematics. Of this I am confident, even though I have read the book attentively (and pleasurably) rather than critically. I found little to criticize in the statement of fact—perhaps nothing that would not lay me open, in view of the book's purpose, to the charge of quibbling.* He has presented these conceptions with a wealth of illustration, and in a style that is always pleasing and often of rare beauty and power. He has developed many and often surprising connections and analogies with apparently remote fields of inquiry. I venture to say that no one, be he professional mathematician or educated layman, can read this book without feeling its stimulating and thought-provoking character, provided only he be philosophically minded. A man not interested in meditating on the general aspect of things would perhaps find the book dull. But what a lot of the joy of life such a man must miss.

J. W. YOUNG

TWO TRANSLATIONS OF ARCHIMEDES

Les Œuvres Complètes d'Archimède. Traduites du Grec en Français avec une introduction et des notes. By Paul Ver Eecke. Paris and Brussels, Desclée, de Brouwer et Cie., 1921. lx + 553 pp.

Kugel und Zylinder von Archimedes. Uebersetzt und mit Anmerkungen versehen. By Arthur Czwalina-Allenstein. No. 202 of Ostwald's Klassiker der exakten Wissenschaften. Leipzig, Akademische Verlagsgesellschaft, 1922. 80 pp.

In considering these two recent evidences of Belgian and German scholarship it may naturally be asked why a new edition of the complete works of Archimedes, or even of a single treatise, should be thought worthy of publication at this time, particularly in view of the fact that we already have the monumental edition by Heiberg, with its recent revision; the

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* In the interest of removing minor blemishes in a future edition, attention may be called to the following: In the group definition of the geometry of shape on p. 218 reference should be to "each and all the transformations of the similitude group" and *no others*; on p. 267, line 7 from the bottom, after the word "field" the restriction ($n \neq n'$) should be added; on p. 329 the statement that a plane of circles is "as rich in circles as in point-triads, as rich in circles as ordinary space in points" is erroneous unless the point-triads be restricted to those formed from points of a line, and is open to misunderstanding, since it leads rather easily to the erroneous idea that dimensionality is a function of the cardinal number of a class rather than of the arrangement of its elements. The extended treatment of the concept of limit seems to me unnecessarily involved and difficult; this portion of the book is hard reading even for one familiar with the concept.

Very few typographical errors were noticed. These occur on p. 136, line 5 from the bottom; on p. 175, line 4 from the bottom; on p. 243 lines 9 and 10; on p. 271, line 11; and on p. 377, line 12.

English edition, with a biographical study, by Heath; the French edition of Peyrard; the German translation of Nizze, and numerous other translations, more or less complete, by such earlier writers as Tartaglia, Venatorius (Thomas Jaeger), Commandinus, Revault (Rivaltus), Maurolycus, and Torelli.

There are various legitimate answers to the question, depending upon the point of view implied. In the first place, the older translators were usually limited to a single manuscript of each work, and that not a very reliable one; textual criticism, in the modern sense, was unknown. In the second place, Heiberg himself, the greatest student of Greek mathematical manuscripts who has undertaken to give a definitive text of Archimedes, based upon the most trustworthy codices, found that the discoveries of Papadopoulos Kerameus, in 1899, necessitated an extended revision of his own edition, and this revision was published in 1913–1915. It follows that all of the earlier translations must now be looked upon as obsolete, if we wish for correct versions. The significance of this statement may be appreciated by comparing the text of Peyrard (Paris, 1807), for example, with that of Heiberg or with the ones under review. Peyrard gives as the first proposition of the *Sphere and Cylinder* a statement which belongs with the fifth postulate of Archimedes, the second proposition being the first as set forth by modern critical writers. Similar variations will be found throughout all of the older editions, but naturally the greatest changes are in the *Stomachion*, the *Method*, and the *Floating Bodies*, all of which have come to light in whole or in newly discovered portions within the last twenty years.

The third justification for a new translation is seen in the general method employed in setting forth the mathematical works of the ancients. Just how far should a translation be literal, and how far should it seek to preserve the form in which the original is cast? The answer to this question depends upon the purpose of the translator. In Heiberg's version he gives the Greek text as it appears in the best manuscripts extant, using the modern forms of the letters and the alphabetic numerals. The text gives, therefore, a correct idea of the original form of statement. In his Latin translation, however, he breaks away from this original form and introduces such modern symbols as appear in the lines

$$\Delta\Gamma + \Gamma B > \Delta B,$$
$$EH : HZ = A\Gamma : A\Theta,$$

and

$$\angle NH\Gamma < 2\,\Lambda KM.$$

When Heath prepared his edition he did the same; but he went even further, arranging the proofs still more after the manner of the modern textbook maker.

As to which of these methods is to be preferred, the reader's preference must be final. The Heath plan is unquestionably the easier for one who is simply searching for the mathematical principles involved; but the method used by Heiberg in the Greek text is the better if one wishes to attune his mind to that of the Greek scholar. Most students who have seriously studied Euclid and Apollonius and Archimedes, for example marvel no doubt at the clarity of the reasoning as set forth in rhetorical form, and at the ability of the Greek mind to accomplish, with no special mathematical symbolism, the results that appear in the classical works of these men. For such students the text which most nearly represents the original is the better; but for the general reader, who is not particularly interested in the mental processes of the Greek, the modern arrangement of proofs, with our common symbols, will be the more simple.

In the two books under review the reader will find the original Greek method set forth at its best in the translation of M. Ver Eecke. It is a model of elegant French and of close rendition of the original text. The scholarly world is greatly indebted to him and to his publishers for their "labor of love." A comparison of the French text with random selections from the Heiberg version shows that the translation has been made with elegance and precision.

The version of Dr. Czwalina-Allenstein is less pretentious, being limited to the *Sphere and Cylinder*. It consists of a careful translation of Heiberg's Latin text, with the modern symbols there used. Instead, however, of displaying the equations and inequalities so that the eye grasps them easily, the translator has run them in with the text. This was doubtless due to the desire to save space, although it is not probable that the size of the book is thereby reduced by more than a couple of pages. The result is that we have neither the original form of expression as seen in the Greek version of Heiberg and in M. Ver Eecke's translation, nor the modern display arrangement as seen in Heath's edition. Nevertheless we have every reason to be thankful for such an accurate piece of work and for its publication in a form that makes the *Sphere and Cylinder* available at a nominal price for all German readers. Indeed, it is with some feeling of envy that one considers the advantages that German scholars have over us in the possession of the Ostwald *Klassiker* in such worthy translations.

On the whole, each of the books under review serves a worthy purpose. The smaller of the works serves such a purpose in that it places in the hands of all who are interested in the subject one of the greatest mathematical classics of the Greeks, and at a price which, even in these troublous times, is insignificant. The more pretentious work of M. Ver Eecke serves a more distinguished purpose, but perhaps no more important one, in that it gives us the first complete modern translation of the works of Archimedes, based upon the latest discoveries and the best textual criticism, and in that this translation gives us the form of the original treatises so that we can more easily place ourselves *en rapport* with the mind of the great genius who gave them to the world.

David Eugene Smith

A COURSE IN EXTERIOR BALLISTICS

A Course in Exterior Ballistics. Ordnance Textbook. Washington, Government Printing Office, 1921. 127 pp.

This is one of the books prepared by the Ordnance Department, and is issued "for the information and guidance of all concerned." It is based upon a course in ballistic methods given in the Ordnance School of Application at the Aberdeen Proving Ground by Captain R. S. Hoar during the winter of 1919–20. Captain Hoar was at that time in charge of the Ballistic Section of the Proof Department at the Proving Ground. He is the real author of the book.

As this is the first attempt to give, in anything like complete form, an exposition of the new ballistic methods developed during the World War in this country, it will be of unusual interest. The criticisms that appear in this review are all intended to be constructive, and are the result of observations made while using the book as one of several texts in a course given at Aberdeen during the next term of the school referred to above.

In Chapters I–IV the author gives what he considers "the irreducible minimum of higher mathematics" necessary for an understanding of the later chapters. As prerequisites the student is supposed to be grounded thoroughly "in algebra, in trigonometry, and to know enough of calculus to appreciate the meaning of a derivative, a differential, and a definite integral." The irreducible minimum consists of partial differentiation, successive approximations, the effect of differential variations, and finite differences. Following Chapter IV the book takes up in succession the following topics: An introduction to modern ballistic methods; the computation of trajectories; the computation of differential corrections; the construction of range tables. Alternate methods and a brief mention of the more involved mathematical processes are given in supplements at the end of the book. These topics will be reviewed in this order.

Of the first four chapters those on successive approximations and finite differences are written clearly and explain these subjects in sufficient detail for a thorough understanding of these methods in subsequent chapters. But in interpolating to find range, time of flight, etc., Lagrange's interpolation formula would serve the purpose quite as well as the one given, is at least as easy to use, and is more easily understood by the average student. The chapter on partial differentiation could have been omitted as it contains nothing which cannot be found in standard texts on calculus with which the student is supposed to have some familiarity. The chapter on differential variations and the subject of differential corrections as later developed are, it seems to me, decidedly inadequate. In my opinion the author should have shown in detail how Major F. R. Moulton first derived the differential equations that must be satisfied by the differences between the coordinates of a point on the standard and those of a point on the disturbed trajectory, how these were subsequently simplified, and how

Professor Bliss introduced his further modification. The introduction of the system of adjoint equations should be presented step by step. To feel that students should have a drill on partial differentiation before undertaking to read the subsequent chapters with appreciation and to assume, tacitly at least, that nothing more would have to be said about adjoint systems than is given in this book, is somewhat out of proportion. One could not ask for a better introduction to this phase on the subject than is given by Professor Dunham Jackson in his ordnance pamphlet published in August, 1919.

The differential equations of the motion of a projectile are

$$X'' = -EX', \qquad Y'' = -EY' - G,$$

where $E = GH/C$, and primes indicate the order of the derivative with respect to t. Of these G is a tabulated function of the *resistance divided by the velocity*, H is a function of the altitude, and C is the ballistic coefficient. These differential equations as well as those used in calculating differential corrections are integrated by means of numerical integration. The introduction of this process as applied to ballistics in this country is due to Major Moulton. The method may be used in this case to obtain any degree of accuracy provided that the physical data are sufficiently accurate. But actually a trajectory so computed is no more accurate than the tabulated G-function, and its accuracy is further limited by the fact that the behavior of a projectile is only approximately characterized by its ballistic coefficient. The newer methods are such a decided advantage, however, over all earlier ones that their acceptance as the best working basis for practical ordnance engineering is not open to question.

It is unfortunate that the tables used in the computation of trajectories and differential corrections are not available to the average reader. They are regarded as confidential by the U. S. Army. To the reader who does not have access to them, these parts of the book will not be more interesting than the solution of problems in spherical trigonometry would be without the proper tables.

One correction that is made unnecessary by the new methods is that for the curvature of the earth. This is accomplished by the introduction of a system of curvilinear coordinates; that is, the distances x are measured along the curved surface of the earth, and the distances y are measured along the extensions of the radii of the earth. In calling attention to the very great superiority of this method of treating the problem it is unfortunate that the author makes a grave error at the bottom of page 67 in stating that the error due to disregarding the curvature of the earth (that is, by regarding the earth as flat) is the enormous quantity $.078\, R^2 \operatorname{ctn} \omega$, where R is the radius of the earth and ω is the angle of fall. The correct expression is $.000000078\, X^2 \operatorname{ctn} \omega$, where X is the range in meters and ω is the angle of fall.

This error shows that it might have been of considerable advantage to work out a numerical illustration of each correction formula. Such an error would not have occurred if that had been done. Also practical persons like to know how much error is to be expected from a given source.

As another instance of the same thing in Chapter XIII a very interesting treatment of the influence of the rotation of the earth upon a projectile leaves the reader in doubt as to whether he should expect a variation of 50 m. or of 150 m. in a given range.

The most recent investigations in ballistics seem to indicate that the effect of yaw has been underestimated. The author mentions the fact that this has usually been allowed for in the ballistic coefficient and that we have only very inaccurate information on the subject at present. Not only is this true, but very slight inaccuracies in the manufacture of two projectiles might be sufficient to cause yaws in different directions of such magnitude that more variation would be produced from this cause than by disregarding entirely corrections which have been laboriously made.

The methods developed in this book, with some variations and additions, have been used by Dr. A. A. Bennett in the construction of new ballistic tables. It was my privilege to use these in the course at Aberdeen. They are easy to understand, easy to use, and from every standpoint are all that could be desired at the present time by the practical ballistician.

Although it is generally conceded that real and lasting contributions to ballistics are made rarely, the use of numerical integration will always stand out as a definite contribution of this period of interest in the subject. Also a very decided advance has been made in the way differential equations have been used to attack problems in ballistics. The book under review is the pioneer in presenting these methods "for the information and guidance of all concerned," and all subsequent books will follow it to a great extent. It is to be hoped that a revision of the book will soon be made, if for no other reason, because of the large number of typographical errors. In spite of the very greatest industry and patience on the part of the author a number of these are still to be found, and the text is manifestly unfair to the author in its present form.

It is not an easy matter to carry through successfully the publication of such a book, especially if one is an army officer and subject to the commands of a superior who may or may not appreciate the character and importance of the work. The Army is fortunate, I think, in having had a man of the author's training, experience, enthusiasm, and ability to get things done, to undertake and carry through such an important contribution to the science of ballistics. Full credit is given to all of the mathematicians whose work forms a part of the text, and it would be unappreciative indeed if full credit were not given to the author for any good result that the publication of the book has had or will have in the future. No course in ballistics would be complete without its use as a text, but it should be used to supplement the old ballistics of Ingalls rather than to replace it. The book cannot be read with facility by the average graduate of West Point because he is unfamiliar with much of the mathematics. It might have the very excellent result of raising the standard of mathematics as taught at West Point.

J. E. ROWE

SHORTER NOTICES

Leçons sur l'Intégration des Équations aux Dérivées Partielles du Premier Ordre. 2d edition, revised and enlarged. By Edouard Goursat. Paris, J. Hermann, 1922. 459 pp.

Almost a generation has passed since the first edition of this book appeared in 1890. It was not reviewed in the BULLETIN, for the BULLETIN itself had not yet appeared above the mathematical horizon. But during all of this time it has been an invaluable work of reference. The first edition was worked out by the author's pupil C. Bourlet from a course of lectures given by Goursat. The present edition has been entirely rewritten by Goursat himself. While the general outline and scope of the book has not been altered, the chapters dealing with existence theorems and linear equations have been entirely remodelled. All of the other chapters have been revised, and though they contain much that is the same as in the first edition, they have been somewhat rearranged and enlarged so that the present volume contains 459 pages as against 354 pages in the first edition.

A comparison of the two editions is very interesting. One finds in the present edition a maturity of thought, a ripeness of judgment, and a touch of the master, which is not present in the first edition. Nevertheless there is a freshness of style about the first edition, and a simplicity that might well recommend it to a younger student. In the preface to this edition Goursat states that he has not taken up the method of Pfaff as he intends to devote an entire volume to the *Problem of Pfaff*, which he hopes will appear soon.

The paper in the copy which we have for review leaves much that is to be desired. It is of poor quality, dark in color, and not of a uniform tint. It reflects, doubtless, the present economic situation.

W. D. MACMILLAN

Darstellende Geometrie. By Th. Schmid, associate professor of geometry at the technical school of Vienna. Volume I, third edition. (Sammlung Schubert, LXV.) Berlin and Leipzig, Vereinigung wissenschaftlicher Verleger, 1922. 283 pages and 170 figures.

The large demand for this popular book has necessitated a third edition of the first volume within three years after the appearance of the second. The characteristic features have been retained, and five new paragraphs added; these are a method of measuring the eccentricity of a conic from a skeleton drawing, its use in putting a cone of revolution through the conic, an approximate construction of π, a determination of the pitch of a regular helix, and a discussion of the existence of an isolated bitangent in the projection of a shadow on a cylinder. Several more historical remarks have been added, and practically all the few typographical errors of the earlier editions corrected.

VIRGIL SNYDER

Elastizitätslehre für Ingenieure, I. *Grundlagen und Allgemeines über Spannungszustände, Zylinder, Ebene Platten, Torsion, Gekrümmte Träger*. By Prof. Dr.-Ing. Max Ensslin. (Sammlung Göschen, Nr. 519.) Zweite Auflage. Berlin und Leipzig, Vereinigung wissenschaftlicher Verleger, 1921. 146 pp.

This little book forms one of that admirable collection of booklets known under the general title of SAMMLUNG GÖSCHEN, and it is a splendid representative of that collection. The problem of the technical theory of elasticity is to show how the forces acting in a given case determine the materials and the form of engineering construction, and to give the means of measurement whereby a breakage is avoided, material is saved, and the form of the unit constructed is held within allowable limits. Engineers have sought to meet the problem partly by a study of experimental data, and they have devised simple approximation formulas and simple graphic methods sufficiently accurate to meet the demands for practical safety and for a reasonable cost of construction. Naturally, the technician soon feels the need of a more accurate method of computing the form and the weight of his materials and the cost of construction; he also desires a proof of his approximation theory. He finds these needs met by the study of the mathematical theory of elasticity; for not only does this latter theory give him valuable insight into the fundamental principles of stresses and strains due to the presence of forces, and into the conditions for the occurrence of the elastic limit, but it also furnishes him many simple formulas useful in computation.

It is the purpose of this and its companion volume to serve as an introduction into these fields. The present volume treats of the general theory of stresses, cylinders, flat plates, the torsion of straight rods, and the deformation of curved rods. Keeping in mind the needs of the technician, the author has applied the theoretical results to many special cases arising in practice, has worked many numerical examples, and has given the graphic representation of the stresses involved in his problems.

JOSEPH LIPKA

Business and Investment Forecasting. By Ray Vance. New York, Brookmire Economic Service, 1922. 132 pp.

A small paragraph must suffice for this modest, unpretentious, but very delightful book. Nowadays business and investment barometers built upon the methods of correlation are not unknown to those who have no idea of scientific statistical methods. Vance's aim is to give the intelligent but not technically informed reader a general notion of the history of such barometers and of the principles of their construction. He appreciates and explicitly points out the great difficulties and uncertainties connected with statistical forecasting, the need for constant revision, for wariness, and for hard study. The main emphasis is naturally laid on the Brookmire Barometers, but there are no hard words for others. Anybody who will read Mr. Vance's little book will gain not only an insight into its subject matter, but will feel a high degree of confidence in the integrity of any barometer constructed by the author.

EDWIN B. WILSON

Einführung in die Theorie der Gewöhnlichen Differentialgleichungen auf Funktionentheoretischer Grundlage. By Ludwig Schlesinger. Dritte, neubearbeitete Auflage. Berlin, Vereinigung wissenschaftlicher Verleger, 1922. 8 + 326 pp.

The first edition of this work appeared in 1900 under the title *Einführung in die Theorie der Differentialgleichungen mit einer unabhängigen Variablen,* and was reviewed for the BULLETIN by Maxime Bôcher. The second edition appeared in 1903 and was merely a reprint of the first. The present edition, however, is thoroughly revised and enlarged, and merits the new title which it has received, for even the first edition was written from the point of view of the theory of functions of a complex variable.

Scarcely half of the original work reappears in the present edition, and even that has been rearranged and more thoroughly germanized by the substitution of words of purely germanic origin for words of a latin origin For example, *Problem* is erased and *Aufgabe* is substituted, *determinieren* is replaced by *bestimmen, Relationen* by *Beziehungen,* etc. This part includes existence theorems for integrals of differential equations of the first order and the discussion of Gauss' and Bessel's equations.

The introduction of the original edition has been expanded into a chapter interpreting the differential equation for real values of the variables. The discussion of linear equations has been simplified by the use of the calculus of matrices. There is a good discussion of differential equations of order higher than the first with fixed critical points, and in the last chapter even the linear integral equations have been touched upon.

All writers upon differential equations insist upon the importance of their subject from the point of view of the applications, and the present writer is no exception. It seems strange therefore that all of them should maintain such a profound silence upon the method, generally known as mechanical quadrature, of following a particular solution in numerical cases. It has been in use, particularly among the astronomers, for a hundred years, and in many cases gives us all of the information which we desire when analytic methods fail. It is much as though writers on the theory of algebraic equations should conspire to ignore Horner's Method and other processes which show us how to find the roots in numerical cases. From the point of view of applications, it merits more attention.

W. D. MACMILLAN

La Matière et l'Énergie selon la Théorie de la Relativité et la Théorie des Quanta. By Louis Rougier. Nouvelle édition, revue et augmentée. Paris, Gauthier-Villars, 1921. 112 pp.

A notice of an English translation, by M. Masius, of the earlier edition of this book has appeared in this BULLETIN (vol. 28, p. 319). As the translator had access to the revised version of the first edition, it is, essentially, the same book. The author appears too ready to accept extreme views with respect both to the theory of relativity and the quantum theory, without subjecting them to a critical survey. On the other hand, he has given a very lucid account of the way in which our conceptions of the interaction of matter and radiant energy have been undergoing a change.

E. P. ADAMS

NOTES

The concluding number of volume 22 of the TRANSACTIONS OF THIS SOCIETY (October, 1921, published in May, 1923) contains the following papers: *On certain numerical invariants of algebraic varieties with application to Abelian varieties* (*continued*), by S. Lefschetz; *On restricted systems of higher indeterminate equations*, by E. T. Bell; *Maximum modulus of some expressions of limited analytic functions*, by S. Kakeya; *Differential variations in ballistics, with applications to the qualitative properties of the trajectory*, by T. H. Gronwall; *An expansion theorem for a system of linear differential equations of the first order*, by W. A. Hurwitz.

The concluding number of volume 44 of the AMERICAN JOURNAL OF MATHEMATICS (October, 1922) contains: *On the kernel of the Stieltjes integral corresponding to a completely continuous transformation*, by C. A. Fischer; *Equivalence and reduction of pairs of Hermitian forms*, by M. I. Logsdon; *Plane cubics with a given quadrangle of inflexions*, by B. M. Turner; *Normal ternary continued fraction expansions for the cube roots of integers*, by P. H. Daus; *Concerning compact Kürschák fields*, by V. D. Gokhale.

The third number of volume 23, series 2, of the ANNALS OF MATHEMATICS (March, 1922) contains: *Dirichlet's problem*, by G. E. Raynor; *Annihilators of modular invariants and covariants*, by O. C. Hazlett; *Systems of linear inequalities*, by W. B. Carver; *Euler squares*, by H. F. MacNeish; *Geometric aspects of Einstein's theory*, by J. Pierpont; *Cauchy's paper of* 1814 *on definite integrals*, by H. J. Ettlinger; *Arithmetical deduction of Kronecker's class-number relations*, by G. H. Cresse; *Cyclotomic heptasection for the prime* 43, by P. O. Upadhyaya; *Summation of a double series*, by T. H. Gronwall.

The following members of the Society have been requested to represent the Committee on Endowment in their respective institutions and in the surrounding territory: C. H. Ashton, Harry Bateman, E. T. Bell, H. F. Blichfeldt, W. C. Brenke, Daniel Buchanan, Paul Capron, D. R. Curtiss, Arnold Dresden, W. H. Echols, John Eiesland, G. C. Evans, J. C. Fields, Tomlinson Fort, Harris Hancock, James Harkness, M. W. Haskell, H. E. Hawkes, E. R. Hedrick, Dunham Jackson, N. J. Lennes, P. H. Linehan, W. R. Longley, J. L. Markley, G. A. Miller, W. E. Milne, Frank Morley, R. K. Morley, G. D. Olds, R. G. D. Richardson, H. L. Rietz, F. H. Safford, G. E. F. Sherwood, H. E. Slaught, C. E. Smith, R. B. Stone, K. D. Swartzel, J. H. Tanner, H. W. Tyler, Oswald Veblen, H. S. White, J. W. Young.

The following important books have been published recently, or are announced for early publication, by American university presses: *Relativity and Modern Physics*, by G. D. Birkhoff, Harvard University Press, 1923; *The Meaning of Relativity*, by Albert Einstein, and *Transformations of Surfaces*, by L. P. Eisenhart, Princeton University Press, 1923 (the latter being published with the cooperation of the National Research Council); *Lectures on Cauchy's Problem in Linear Partial Differential Equations*, by Jacques Hadamard, announced by the Yale University Press for early publication.

The Berlin Academy of Sciences announces the early publication of volume 7 of Weierstrass's Works.

The following honorary doctorates have been conferred: by the University of Edinburgh, on Professor H. F. Baker, of St. John's College, Cambridge; by the University of Aberdeen, on Professor E. W. Hobson, of Christ's College, Cambridge; by the University of Pennsylvania, on Sir Joseph Thomson, master of Trinity College.

The tricentenary of the birth of Blaise Pascal, which occurs on June 19, 1923, will be commemorated by celebrations to be held on July 8–9, under the auspices of the French government and the Paris Academy of Sciences.

Professor A. A. Michelson, of the University of Chicago, has been elected president of the National Academy of Sciences. Professors E. W. Brown, of Yale University, Max Mason, of the University of Wisconsin, and D. L. Webster, of Stanford University, have been elected to membership in the Academy.

Professor A. A. Michelson has been elected first vice-chairman of the National Research Council. Professor M. I. Pupin, of Columbia University, has been elected a member at large of the executive board.

Sir Ernest Rutherford, of Cambridge University, has been elected an honorary foreign member of the Washington Academy of Sciences.

The price of the JAHRBUCH ÜBER DIE FORTSCHRITTE DER MATHEMATIK has been substantially reduced. Beginning with volume 46, the first number of which has just been published, the price will be about one cent a page, with a reduction of twenty-five per cent to members of the American Mathematical Society and the Mathematical Association of America. This reduced price is practically the same as the pre-war price. To take advantage of it, orders should be sent directly to Professor L. Bieberbach, Berlin-Schmargendorf, Marienbaderstrasse 9.

Professor F. S. Carey has resigned from the chair of mathematics at the University of Liverpool, after thirty-seven years of service.

Professor M. J. M. Hill, of University College, London, has resigned.

At the University of Grenoble, M. Gosse has been promoted to a professorship of mathematics.

Dr. H. M. Dadourian, associate professor of physics at Trinity College, Hartford, has been appointed Seabury professor of mathematics.

Dr. Louis Weisner, of Columbia University, has been appointed instructor in mathematics at the University of Rochester.

Mr. J. B. Rosenbach, instructor in mathematics at the Carnegie Institute of Technology, has been made assistant professor of mathematics.

Mr. O. H. Rechard, Jr., instructor in mathematics at the University of Wisconsin, has been appointed assistant professor of mathematics at the University of Wyoming.

Professor J. L. Riley has resigned his professorship of mathematics at John Tarleton Agricultural College, Stephenville, Texas.

At Rutgers College, Professor A. A. Titsworth, formerly in the department of Civil Engineering, has been transferred to the department of mathematics as a full professor. Mr. W. B. Campbell has been appointed instructor.

Miss Ruth Thompson has been appointed instructor in mathematics at the New Jersey College for Women.

The following appointments have been made in the Department of Mathematics at Harvard: Philip Franklin, Benjamin Peirce instructor; H. L. Garabedian, K. W. Halbert, R. L. Jeffrey, E. R. C. Miles, F. W. Perkins, M. M. Slotnick and T. L. Smith, instructors.

At Ohio Northern University, Mr. J. T. Fairchild has been promoted to a professorship of mathematics.

Rev. Paul Muehlman, formerly head of the department of mathematics at Marquette University, has been transferred to Loyola University, Chicago, as head of the department there.

Mr. B. F. Kimball has been appointed instructor in mathematics at Tulane University.

At the Texas Agricultural and Mechanical College, Assistant Professors D. C. Jones and W. L. Porter have been promoted to associate professorships of mathematics, Mr. W. L. Hughes and Mr. P. K. Smith have been appointed assistant professors, and Mr. C. E. McCurry an instructor.

At the University of Denver, Professor G. W. Gorrell of the Colorado State School of Mines has been appointed an associate professor, and Dr. E. Frances Seiler of the University of Illinois an instructor.

Professor Matyáš Lerch, formerly of the University of Freiburg, Switzerland, and since 1906 of the Polytechnic Institute at Brno, died August 3, 1922, at the age of sixty-two. The Academy of Sciences of the Institute of France awarded to Professor Lerch in 1900 the grand prize for mathematical sciences.

Professor Čenek Jarolímek, of the Polytechnic Institute of Prague, died December 14, 1921, at the age of seventy-five.

Professor J. D. van der Waals, of the University of Amsterdam, died March 8, 1923, at the age of eighty-five.

Dr. J. G. Leathem, fellow and bursar of St. John's College, Cambridge, coeditor with Professor E. T. Whittaker of the Cambridge Mathematical Tracts, died March 19, 1923, at the age of fifty-one.

Sir John Venn, president of Gonville and Caius College and for many years lecturer in logic and philosophy at Cambridge University, died April 4, 1923, at the age of eighty-eight.

Professor A. G. Webster, of Clark University, died May 15, 1923, at the age of fifty-nine. He had been a member of this Society since 1891.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

Boehm (K.). Begriffsbildung. Karlsruhe, G. Braun, 1922. 46 pp.

Creak (T. G.). See Cunningham (A.).

Cresse (G. H.). See Dickson (L. E.).

Cunningham (A.), Woodall (H. J.) and Creak (T. G.). Haupt-exponents, primitive roots, and standard congruences. London, F. Hodgson, 1922. 136 pp.

Dickson (L. E.). History of the theory of numbers. Volume III: Quadratic and higher forms. With a chapter on the class number, by G. H. Cresse. Washington, Carnegie Institution, 1923. 8vo. 4 + 313 pp.

Eisenhart (L. P.). Transformations of surfaces. Published with the cooperation of the National Research Council. Princeton, University Press, 1923. 10 + 380 pp. $4.00

Epstein (P.). See Weber (H.).

Fettweis (E.). Wie man einstens rechnete. (Mathematisch-Physikalische Bibliothek, Nr. 49.) Leipzig, Teubner, 1923. 56 pp.

Junker (F.). Höhere Analysis. I: Differentialrechnung. 3te, verbesserte Auflage. (Sammlung Göschen.) Berlin, Vereinigung wissenschaftlicher Verleger, 1921. 8vo. 204 pp.

Miner (J. R.). Tables of $\sqrt{1 - r^2}$ and $1 - r^2$ for use in partial correlation and in trigonometry. Baltimore, Johns Hopkins Press, 1922. 8vo. 49 pp.

Müller (C.). Mathematik. In Vom Altertum zur Gegenwart. Die Kulturzusammenhänge in den Hauptepochen und auf den Hauptgebieten. Skizzen. 2te, vermehrte Auflage. Leipzig, Teubner, 1921. 10 + 386 pp.

Otzen (R.). See Timerding (H. E.).

Schiller (L.). Untersuchungen über laminare und turbulente Strömung. (Leipziger Habilitationsarbeit.) Berlin, Verlag des Vereines deutscher Ingenieure, 1922. 36 pp.

Steele (R.). The earliest arithmetics in English. Edited, with an introduction, by R. Steele. London, Oxford University Press, 1923. 8vo. 18 + 84 pp.

Timerding (H. E.). Mathematik. (Handbibliothek für Bauingenieure, herausgegeben von R. Otzen, Band I, Teil I.) Berlin, Springer, 1923. 8 + 242 pp.

Weber (H.) und Wellstein (J.). Enzyklopädie der Elementarmathematik. 1ter Band: Arithmetik, Algebra und Analysis. 4te Auflage, neubearbeitet von Paul Epstein. Leipzig, Teubner, 1922. 16 + 568 pp.

Wellstein (J.). See Weber (H.).

Woodall (H. J.). See Cunningham (A.).

PART II. APPLIED MATHEMATICS

ALLIATTA (G.). Die Radioaktivität im Weltbild der Aethermechanik. Leipzig, Hillmann, 1922. 8vo. 14 pp.

ANDOYER (H.). Cours de mécanique céleste. Tome I. Paris, Gauthier-Villars, 1923.

BIRKHOFF (G. D.). Relativity and modern physics. Cambridge, Mass., Harvard University Press, 1923.

BOHR (N.). Atomernes Bygning og Stoffernes fysiske og kemiske Egenskaber. Kjøbenhavn, Gjellerup, 1922. 72 pp.

BORN (M.). La théorie de la relativité d'Einstein et ses bases physiques. Traduit sur la 2e édition allemande par F. A. Finkelstein et J. G. Verdier. Paris, Gauthier-Villars, 1923. 8vo. 339 pp.

BOTTLINGER (K. F.). See STRÖMGREN (E.).

BOUTARIC (A.). La vie des atomes. Paris, Flammarion, 1923. 16mo. 248 pp.

BRILLOUIN (L.). La théorie des quanta et l'atome de Bohr. Paris, Les Presses Universitaires de France, 1922. 8vo. 181 pp.

———. See FABRE (L.)

DE BROGLIE (M.). Les rayons X. Paris, Les Presses Universitaires de France, 1922. 8vo. 164 pp.

BURTON (E. F.). The physical properties of colloidal solutions. 2d edition. London, Longmans, 1921. 8vo. 220 pp.

COURANT (R.). See MADELUNG (E.).

CURRY (W. A.). See MORECROFT (J. H.).

DINWOODIE (W.). Wave power transmission. London, Spon, 1923. 8vo. 28 pp.

DREYER (G.). Formeln, Begriffserklärungen und Lehrsätze aus der reinen und angewandten Festigkeitslehre. 3te Auflage. Leipzig, Jänecke, 1922. 8 + 96 pp.

FABRE (L.). Les théories d'Einstein. Nouvelle édition, épurée, accrue de notes liminaires, d'un exposé des théories de Weyl, et de trois notes de MM. Guillaume, Brillouin, et Sagnac sur leurs propres idées. Paris, Payot, 1922. 8vo. 255 pp.

FICHOT (E.). Les marées et leur utilisation industrielle. Paris, Gauthier-Villars, 1922. 8vo. 256 pp.

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FISCHER (V.). Eine Darstellung des Nernst'schen Wärmetheorems. Teil 1. Frankfurt, Blazek und Bergmann, 1923. 8vo. 23 pp.

FLEURY (H.). Exposé élémentaire de la théorie d'Einstein. Paris, E. Larose, 1922. 8vo. 34 pp.

FONTENÉ (G.). La relativité restreinte, avec un appendice sur la relativité géneralisée. Paris, Vuibert, 1923. 12mo. 116 pp.

FOOTE (P. D.) and MOHLER (F. L.). The origin of spectra. (American Chemical Society Monograph Series.) New York, Chemical Catalogue Company, 1922. 8vo. 250 pp.

FUCHS (F.). Elektrische Strahlen und ihre Anwendung (Röntgentechnik). München und Berlin, Oldenbourg, 1922. 35 pp.

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MAREC (E.). Les enroulements industriels des machines à courant continu et à courants alternatifs. Théorie et pratique. 2e édition. Paris, Gauthier-Villars, 1921. 8vo. 9 + 240 pp.

VON MISES (R.). Fluglehre. 2te, durchgesehene Auflage. Berlin, Springer, 1922. 8 + 210 pp.

MOHLER (F. L.). See FOOTE (P. D.).

MORECROFT (J. H.), PINTO (A.) and CURRY (W. A.). Principles of radiocommunication. New York, Wiley, and London, Chapman and Hall, 1921. 10 + 935 pp.

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SCHMIDT (H.). Das Weltbild der Relativitätstheorie. 3te Auflage. Hamburg, Hartung, 1922. 8vo. 8 + 139 pp.

SCHWINGE (O.). Eine Lücke in der Terminologie der Einsteinschen Relativitätslehre. Berlin, Commissionsverlag Puhle, 1921. 30 pp.

STEPHAN (P.). Die technische Mechanik des Maschineningeneurs. Band 1: Allgemeine Statik. Band 2: Die Statik der Maschinenteile. Band 3: Bewegungslehre und Dynamik fester Körper. Berlin, 1921–22.

STOCK (A.). The structure of atoms. Translated from the second German edition by S. Sugden. Revised and enlarged. London, Methuen, 1923. 8 + 88 pp.

STRÖMGREN (E.). Astronomische Miniaturen. Aus dem Schwedischen übersetzt von K. F. Bottlinger. Berlin, Springer, 1922. 8 + 88 pp.

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WALDEN (P.). Optische Umkehrerscheinungen (Waldensche Umkehrung). Braunschweig, Vieweg, 1919. 5 + 214 pp.

WARNANT (L.). Les théories d'Einstein. Essai de réfutation. Examen critique. Paris, Alcan, 1922. 12mo. 144 pp.

REPORT ON CONTINUOUS CURVES FROM THE VIEWPOINT OF ANALYSIS SITUS*

BY R. L. MOORE

1. *The Characterization of a Continuous Curve.* A *continuous curve* is the set of points represented by the pair of equations

$$x = f_1(t), \qquad y = f_2(t),$$

where $f_1(t)$ and $f_2(t)$ are continuous functions of t on the interval $I : (0 \leqq t \leqq 1)$. Thus a continuous curve is the image of the straight line interval I under a continuous † transformation which transforms each point X of the interval I into a single point $T(X)$ of the curve. In case this transformation does not throw any two distinct points of I into the same point of the curve $T(I)$, then $T(I)$ is a *simple continuous arc.* Thus a simple continuous arc is in one to one continuous correspondence ‡ with a straight line interval. If $T(0) = T(1)$, but no point of $T(I)$ except $T(0)$ is the transform, under T, of more than one point of I, then $T(I)$ is a *simple closed curve.* It easily follows that a simple closed curve is in one to one continuous correspondence with a circle.

It is clear from the above definition that every continuous curve is a bounded continuum.* That it is not, however, very easy to determine directly from this definition whether or not even some of the simplest continua are continuous curves will, I think, be admitted if one recalls the interest aroused by Peano's discovery of the fact that a square plus its interior is such a curve and the proofs of this fact that were given by Peano, Hilbert and E. H. Moore.†

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* Presented before the Southwestern Section of the Society, in somewhat different form, at the Symposium held in Lawrence, Kansas, Dec. 2, 1922.

† The point P is said to be a *limit point* of the point set M if, for every positive number ϵ, there are points of M, distinct from P, at a distance from P less than ϵ. A transformation T which throws a point set M into a point set $T(M)$ is said to be *continuous* if, in case the point P of M is a limit point of a point set N which is a subset of M, then $T(P)$ is a limit point of $T(N)$.

‡ Two point sets M and N are said to be in *one to one correspondence* if there exists a correspondence in which (*a*) to each point P of M there corresponds just one point P' of N, (*b*) no two distinct points of M correspond to the same point of N, (*c*) if the point P' of N corresponds to the point P of M, then the point P of M corresponds to the point P' of N and conversely. Such a correspondence is sometimes called a one to one *reciprocal* correspondence. For so-called one to one correspondences which are not reciprocal I prefer to use the term transformation which seems to me to be more suggestive of an operation which is thought of as taking place in one direction.

In 1908, Schoenflies established a theorem which embodies a point set theoretic characterization (definition) of a continuous curve. This is, apart from minor matters of phraseology, etc., as follows: In order that a bounded continuum M should be a continuous curve it is necessary and sufficient that (*a*) for every given positive number ϵ there are not more than a finite number of complementary domains * of M of diameter greater than ϵ, (*b*) the boundary of every complementary domain of M is accessible † at each of its points, from all sides, with respect to that domain.

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* A set of points is said to be *closed* if it contains all its limit points. A point set is said to be *connected* if, however it be divided into two mutually exclusive subsets, one of them contains a limit point of the other one. (Cf. N. J. Lennes, American Journal, vol. 33 (1911), pp. 287–326.) A set of points is said to be *bounded* if it lies wholly in the interior of some circle. A *continuum* is a closed and connected point set.

† Peano, *Sur une courbe, qui remplit toute une aire plane,* Mathematische Annalen, vol. 36 (1890); Hilbert, *Ueber die stetige Abbildung einer Linie auf ein Flächenstück*, ibid., vol. 38 (1891), pp. 459–460; E. H. Moore, *On certain crinkly curves*, Transactions of this Society, vol. 1 (1900), pp. 72–90. That a square plus its interior is not a simple continuous arc had been proved by Netto about ten years before. (E. Netto, *Beitrag zur Mannigfaltigkeitslehre*, Journal für Mathematik, vol. 86 (1879), pp. 263–268.) On page 329 of the 1907 edition of his *The Theory of Functions of a Real Variable*, Hobson gives a proof of this theorem. Referring to the proofs of Netto and of Loria, he says: "In the proof given by these writers it is assumed that a closed curve corresponds to a linear sub-interval of (0, 1); this is not necessarily the case, for a non-dense closed set may correspond to the closed curve." This statement of Hobson's (which is repeated in the 1921 edition of his treatise) is manifestly incorrect. Neither a simple closed curve nor any other sort of connected point set could possibly be in one to one continuous correspondence with a non-dense closed set or any other set which is not connected. As far as I know, I have never seen anywhere in the literature the following extremely simple proof of Netto's theorem. Let X denote a point, other than an end-point, of the interval I. Suppose there exists a continuous one to one correspondence between I and the point set K constituted by a square plus its interior. Let X' denote the point of K which corresponds to X. The set $I - X$ is not connected. Hence $K - X'$ is not connected. But clearly $K - X'$ *is* connected. Thus the supposition that the correspondence in question exists leads to a contradiction. The same proof shows that there is no one to one continuous correspondence between the interval I and the set of points constituted by a simple closed curve.

That the same class of point sets would not be characterized by this definition if accessibility from all sides were replaced by mere accessibility may be seen with the help of the following example. Let M_1 denote the point set composed of the straight line intervals AB, BC, CD, where the coordinates of the points A, B, C, and D are $(0, 1)$, $(0, -2)$, $(1/\pi, -2)$ and $(1/\pi, 0)$ respectively. Let M_2 denote that portion of the curve $y = \sin (1/x)$ which lies between the lines $x = 0$ and $x = 1/\pi$. Let M_3 denote the point set $M_1 + M_2$. The point set M_3 separates the plane into two domains K and H, where H is bounded. Let M denote the continuum $H + M_3$. The only domain complementary to M is K. Furthermore M_3, the boundary of K, is accessible at each of its points with respect to K. That it is not however accessible at every point from all sides with respect to K may be seen as follows. Let E denote the point $(1/\pi, 2)$ and let AED denote the simple continuous arc composed of the straight line intervals AE and ED. The origin O is not accessible from both of the domains into which K is divided by the point set $AED - (A + D)$, although O lies on the boundary of each of these domains. Thus O is not accessible from all sides with respect to K. Hence, by Schoenflies' theorem, M is not a continuous curve.

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* A connected point set K is said to be a domain if for each point P of K there exists a positive number δ_P such that K contains every point whose distance from P is less than δ_P. The boundary of a point set K is the point set composed of all points $[X]$ such that for each positive number ϵ there is at least one point of K and at least one point which does not belong to K, at a distance from X less than ϵ. A complementary domain of a closed point set M is a domain K which contains no point of M but whose boundary is a subset of M. The diameter of a bounded point set K is a number d such that (*a*) if X and Y are any two points of K then the distance from X to Y is not greater than d, (*b*) for every positive number e there exist points X and Y belonging to K such that the distance from X to Y differs from d by less than e.

† The boundary B of a domain K is said to be accessible at the point P with respect to K (and P is said to be accessible from K) if, for every point A in K, there exists a simple continuous arc AP which lies, except for the point P, wholly in the domain K. Suppose that XYZ is a simple continuous arc whose end-points X and Y lie on B, but every other point of which lies in the domain K. If B is connected then K is simply connected and is separated by the point set $XYZ - (X + Y)$ into two mutually exclusive domains, K_1 and K_2, such that $K = XYZ - (X + Y) + K_1 + K_2$. At least one of these domains (call it D_{XYZ}) has the point P on its boundary. If, for every arc XYZ satisfying the conditions stipulated, the point P is accessible from D_{XYZ}, P is said to be accessible from K from all sides and B is said to be accessible at P, from all sides, with respect to K.

While Schoenflies' definition holds good for continuous curves in the plane, it does not hold for those in three dimensions. In fact I have recently shown * that, in order that, in space of three dimensions, a continuum M should be a continuous curve, it is not sufficient that it should have the two properties, (*a*) and (*b*), stipulated in Schoenflies' definition and it is not necessary that it should have *either* of them.

In 1914 Hans Hahn showed that in order that, in space of any number of dimensions, a bounded continuum should be a continuous curve, it is necessary and sufficient that it should have a property which he designates as that of *connectedness im kleinen.*† A continuum is said to be *connected im kleinen* at the point P if for every positive number ϵ there exists a positive number $\delta_{P\epsilon}$ such that if X and Y are any two points at a distance from P less than $\delta_{P\epsilon}$ then they lie together in a closed and connected subset of M every point of which is at a distance less than ϵ from the point P. This definition of a continuous curve has, for many purposes, a decided advantage over that of Jordan. For instance, it is easy to see that a square plus its interior (or, in three dimensions, a cube plus its interior) satisfies this definition and is therefore a continuous curve. Consider, on the other hand, the point sets M_1, M_2, M_3, described as follows:

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* *On the relation of a continuous curve to its complementary domains in space of three dimensions*, PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 8 (1922), pp. 33–38. On page 38 of this paper there is a theorem, involving conditions concerning the complementary domains of a bounded continuum M, in space of three dimensions, which are sufficient, but not necessary, in order that M should be a continuous curve.

† Hans Hahn, *Über die allgemeinste ebene Punktmenge, die stetiges Bild einer Strecke ist*, JAHRESBERICHT DER VEREINIGUNG, vol. 23 (1914), pp. 318–322. See also, however, S. Mazurkiewicz, *Sur les lignes de Jordan*, FUNDAMENTA MATHEMATICÆ, vol. 1 (1920), pp. 166–209. In this paper Mazurkiewicz introduces the same conception and establishes the same result, and refers, in this connection, to earlier papers of his own as follows: *O arytmetyzacji kontinuow*, C. R. SOC. SC. VARSOVIE, vol. 6 (1913); *O arytmetyzacji kontinuow*, II, ibid., vol. 6 (1913); *O pewnej klasyfikacji punktow lezacych na kontinuach dowolnych*, ibid., vol. 9 (1916). I do not have access to these papers. The conception of *connectedness im kleinen*, as applied to a simple closed curve, was used by Pia Nalli, in the paper *Sopra una definizioni di dominio piano limitato da una curva continua, senza punti multipli*, RENDICONTI DI PALERMO, vol. 32 (1911), pp. 391–401.

Let M_1 denote the point set obtained by adding to the interval from $(0, -1)$ to $(0, 1)$ that part of the curve $y = \sin (1/x)$ which lies to the right of the y-axis and is bounded on the right by the line $x = 1$. Let M_2 denote the point set consisting of the intervals OB, OB_1, OB_2, OB_3, $\cdots$ where O is the origin of coordinates, B is the point $(0, 1)$ and, for every n, B_n is the point $(1/n, 1)$. Let M_3 denote the point set obtained by adding to the set M_1 the set of all the intervals $A_{mn}B_{mn}$ where, for every pair of positive integers m and n, A_{mn} and B_{mn} denote the points $(0, m/n)$ and $(m/n, m/n)$ respectively. I venture to surmise that one who is acquainted with no other characterization of a continuous curve than that of Jordan would probably not be able to decide, without some difficulty, the question, which of the sets M_1, M_2, M_3 is a continuous curve and which are not.* It is easy, however, to see that neither M_1 nor M_2 is connected im kleinen at the point $(0, 1/2)$, while M_3 is connected im kleinen everywhere. The set M_3 is accordingly a continuous curve, while M_1 and M_2 are not.

Mazurkiewicz, Tietze, and I, working, as far as I know, entirely independently of each other, have shown † that, in order that a bounded continuum should be a continuous curve it is necessary that it should be arc-wise * connected. That this condition is not sufficient is shown by the existence of the above-described point set M_2, which, though arc-wise connected, is not a continuous curve. However, I have recently established the following theorem which embodies a generalization of the above-mentioned result.

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* For examples in space of three dimensions see my paper *On the relation of a continuous curve to its complementary domains in space of three dimensions*, loc. cit.

† S. Mazurkiewicz, loc. cit., H. Tietze, *Ueber stetige Kurven, Jordansche Kurvenbogen und geschlossene Jordansche Kurven*, MATHEMATISCHE ZEITSCHRIFT, vol. 5 (1919), pp. 284–291; R. L. Moore, *A theorem concerning*

THEOREM A. *In order that a continuum M should be a continuous curve it is necessary that every maximal connected subset of an open subset of M should be arc-wise connected.*†

Less than a week before the date of the delivery of this address, Mr. R. L. Wilder showed ‡ that this condition is also *sufficient.* It therefore affords a complete characterization of a continuous curve. It may be of interest to see just how this condition fails to hold true for the set M_2. Let C denote the point (0, 1/2). Though the set M_2 is arc-wise connected and $M_2 - C$ is an open connected subset of M_2, the points O and B cannot be joined by a simple arc which is a subset of $M_2 - C$.

Sierpinski has shown § that a continuous curve may be characterized as a bounded continuum M such that, for every positive number ϵ, M is the sum of a finite collection of closed and connected point sets each of which is of diameter less than ϵ.

Thus we have five characterizations of the notion *continuous curve*, that of Jordan (or perhaps it should be called a modifi-

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continuous curves, this BULLETIN, vol. 23 (1917), pp. 233–236. This paper was presented to the American Mathematical Society, October 28, 1916.

* A point set M is said to be arc-wise connected if every two distinct points of M are the extremities of a simple continuous arc which lies wholly in M.

† R. L. Moore, *Concerning continuous curves in the plane*, MATHEMATISCHE ZEITSCHRIFT, vol. 15 (1922), pp. 254–260. The point set K is said to be an open subset of the set M if K is a subset of M and $M - K$ is either vacuous or closed. A *maximal* connected subset of a point set K is a connected subset of K which is not a proper subset of any other subset of K.

‡ This result forms a part of Mr. Wilder's dissertation for the degree of Ph.D. at the University of Texas.

§ W. Sierpinski, *Sur une condition pour qu'un continu soit une courbe jordanienne*, FUNDAMENTA MATHEMATICÆ, vol. 1 (1920), pp. 44–60.

cation of that of Jordan), that of Schoenflies, that of Hahn and Mazurkiewicz, that of Sierpinski, and lastly a characterization whose sufficiency and necessity have been established by Mr. Wilder and by the author, respectively. Of these characterizations, all but that of Schoenflies hold good in space of any number of dimensions.

2. *The Relation of a Continuous Curve to its Complementary Domains, in Space of two Dimensions.* Returning to Schoenflies' characterization, we may picture any continuous curve in the plane as the point set complementary to a finite or countably infinite set of simply connected domains such that (*a*) one of these domains is unbounded and contains, as a subset, the exterior of some circle which encloses the continuous curve, (*b*) for every positive number ϵ there are not more than a finite number of these domains of diameter greater than ϵ, (*c*) the boundary of each of these domains is, at each of its points, accessible from all sides with respect to that domain. Miss M. Torhorst has shown that the boundary of every complementary domain of a continuous curve is itself a continuous curve.* I have recently found that the outer boundary of every bounded complementary domain of a continuous curve is a simple closed curve and that if two points are separated from each other by a continuous curve M then they are also separated by some simple closed curve which is a subset of M.† From this result, with the aid of the fact ‡ that every simple closed curve separates the plane, it follows that in order that a non-dense continuous curve

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* Cf. Marie Torhorst, *Ueber den Rand der einfach zusammenhängenden ebenen Gebiete*, MATHEMATISCHE ZEITSCHRIFT, vol. 9 (1921), pp. 44–65. In his thesis Mr. Wilder has shown that, indeed, every closed and connected subset of such a boundary is a continuous curve. This result extends, to the case of any such boundary, a proposition previously established by Mazurkiewicz for the case of a continuous curve which contains no simple closed curve. Cf. S. Mazurkiewicz, *Un théorème sur les lignes de Jordan*, FUNDAMENTA MATHEMATICÆ, vol. 2 (1921), pp. 119–130.

† *Concerning continuous curves in the plane*, loc. cit.

‡ Cf. O. Veblen, *Theory of plane curves in non-metrical analysis situs*, TRANSACTIONS OF THIS SOCIETY, vol. 6 (1905), pp. 107–112; and L. E. J. Brouwer, *Beweis des Jordanschen Kurvensatzes*, MATHEMATISCHE ANNALEN, vol. 69 (1910), pp. 169–175.

should separate the plane it is necessary and sufficient that it should contain a simple closed curve.

I will say that a point set M has property S if, for every positive number ϵ, M is the sum of a finite number of connected subsets all of diameter less than ϵ. As applied to closed (and bounded) point sets this property is[*] equivalent to that of connectedness im kleinen. But, as applied to point sets which are not necessarily closed, it is[†] stronger than that of connectedness im kleinen and weaker than that of uniform[‡] connectedness im kleinen. Every domain is connected im kleinen. In order that a bounded and simply connected domain should have a continuous curve as its boundary it is[§] necessary and sufficient that it should have property S, while in order that it should have a simple closed curve as its boundary it is necessary and sufficient that it should have the stronger property of being uniformly connected im kleinen.[||]

In his thesis Mr. Wilder makes a detailed study of the continuous curve which is the boundary of a domain. He has found that in order that the boundary B of a simply connected domain should be a continuous curve it is necessary and sufficient that every connected subset of B should be arc-wise connected.

3. *A Characterization of Continua which are not Continuous Curves.* I have found the following theorem very useful in the study of continuous curves.

THEOREM. *In order that a bounded continuum M should fail to be a continuous curve, it is necessary and sufficient that there should exist two concentric circles k_1 and k_2 and a countable*

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[*] Cf. W. Sierpinski, loc. cit.

[†] Cf. my paper, *Concerning continuous curves in the plane*, loc. cit.

[‡] A point set M is said to be uniformly connected im kleinen if, for every positive number ϵ, there exists a positive number δ_ϵ such that if X and Y are two points of M at a distance apart less than δ_ϵ then X and Y lie in some connected subset of M of diameter less than ϵ. Cf. Hans Hahn and S. Mazurkiewicz, loc. cit.

[§] R. L. Moore, loc. cit.

[||] See my paper, *A characterization of Jordan regions by properties having no reference to their boundaries*, PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 4 (1918), pp. 364–370.

infinity of continua $\overline{M}$, M_1, M_2, M_3, $\cdots$, *such that* (1) *each of these continua is a subset of* M *and contains at least one point on* k_1 *and at least one point on* k_2, *and is a subset of the point set* H *which is composed of the two circles* k_1 *and* k_2 *together with all those points of the plane which lie between these circles,* (2) *no two of these continua have a point in common, and, indeed, no one of them is a proper subset of any connected point set which is common to* M *and* H, (3) *the set* $\overline{M}$ *is the sequential limiting set of the sequence of sets* M_1, M_2, M_3, $\cdots$,* (4) *there exists a connected subset of* M *which contains all the sets of the sequence* M_1, M_2, M_3, $\cdots$, *but which contains no point of the greatest connected point set which is common to* $\overline{M}$ *and* H.

For an indication of a proof of all but part (4) of this theorem see my papers, *Continuous sets that have no continuous sets of condensation*, and *A characterization of Jordan regions by properties that have no reference to their boundaries.*† The important part (4) has been supplied recently by Mr. Wilder. With its aid he established the above-mentioned theorem that in order that a bounded continuum M should be a continuous curve it is sufficient that every maximal connected subset of an open subset of M should be arc-wise connected. I have made use of parts (1), (2), and (3) on several occasions.

4. *A Continuous Curve in the Rôle of a Space.* It is of interest to note that if the set of points S which constitute a continuous curve is viewed as a space and certain terms are properly defined, then many of the theorems which hold true in ordinary space continue to hold true in S. For each point

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* The point set M is said to be the limiting set of the sequence of point sets M_1, M_2, M_3, $\cdots$ provided that (*a*) each point of M is the sequential limit point of an infinite subsequence of some sequence of points P_1, P_2, P_3, $\cdots$ such that, for every n, P_n belongs to M_n, (*b*) if P_1, P_2, P_3, $\cdots$ is a sequence of points such that, for every n, P_n belongs to M_n, then M contains the sequential limit point of every subsequence of P_1, P_2, P_3, $\cdots$ that has a sequential limit point. If the further condition is satisfied that every infinite subsequence of the sequence M_1, M_2, M_3, $\cdots$ has the same limiting set M, then M is said to be the sequential limiting set of the sequence M_1, M_2, M_3, $\cdots$.

† This BULLETIN, vol. 25 (1919), pp. 174–176; and PROCEEDINGS OF THE NATIONAL ACADEMY, loc. cit.

P in S and each positive number ϵ, let $R_{P\epsilon}$ denote the set of all those points of S which lie, together with P, in a connected subset of S which lies wholly within a circle with center at P and radius equal to ϵ. For every P and ϵ the point set $R_{P\epsilon}$ will be called a *region with respect to* S. With the use of the fact that S is connected im kleinen, it may be seen that a point X in S is a limit point of a point set M which lies in S if and only if every region (with respect to S) which contains X contains also a point of M distinct from X. It follows that if S is identified with the S of my set Σ_1 of axioms * for plane analysis situs then the point X is a limit point of M in the ordinary sense if and only if it is a limit point of M in the sense defined in F. A. With the help of this fact it may be seen that Axioms 1, 2, and 4, and a number of the theorems of that paper hold true in the space S, the terms (other than region) that are used in the statement of those axioms and theorems being given meanings which are ordinarily attached to them when S is considered as a subset of ordinary space. Thus, corresponding to Theorem 15 of F. A., we have the previously mentioned theorem that every two points of a connected open subset † of a continuous curve can be joined by a simple continuous arc lying wholly in that subset.

In his thesis, in addition to other results in this connection, Mr. Wilder has established the following theorem.

THEOREM. *If S is a continuous curve in space of two dimensions, D is a domain with respect to S, B is the boundary of D with respect to S, and P is a point of D, then in order that P should be accessible, in S, from every point of D it is sufficient either* (*a*) *that B should be connected im kleinen, or* (*b*) *that B should contain no continuum of condensation ‡ which contains P.*

5. *The Non-Cut Points of a Continuous Curve.* A point P of

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* *On the foundations of plane analysis situs*, TRANSACTIONS OF THIS SOCIETY, vol. 17 (1916), pp. 131–164. This paper will be referred to hereafter as F. A.

† A connected open subset of S is called a domain in F. A., and such a subset of a continuous curve S may be called a domain with respect to S.

‡ A sub-continuum K of a point set M is said to be a *continuum of condensation* of M if every point of K is a limit point of $M - K$.

a connected point set M is said to be a *non-cut point* of M or a *cut point* of M according as $M - P$ is or is not connected. A cut point of M is said to disconnect M in the strong sense (or, merely, to disconnect M). A point set M is said to be connected in the strong sense if every two points of M lie together in some closed and connected subset of M. If $M - P$ is not connected in the strong sense then P is said to disconnect M in the weak sense. According to this definition, if a point P disconnects a point set M in the strong sense then it also disconnects it in the weak sense; but the converse is not always true. By Theorem A of § 1, if M is a continuous curve, then it is true, conversely, that if P disconnects M in the weak sense it also disconnects it in the strong sense.

In my paper *Concerning simple continuous curves*,[*] I proved that if a bounded continuum M does not contain more than two non-cut points then it is a simple continuous arc. But a simple continuous arc contains two non-cut points, namely its end-points. Thus we have the following theorem.

THEOREM C. *Every bounded continuum contains at least two non-cut points (i.e., two points neither of which disconnects it).*[†]

In his paper, *Un théorème sur les lignes de Jordan*,[‡] Mazurkiewicz has established the following theorems.

THEOREM $\overline{\text{C}}$. *Every continuous curve contains at least two points neither of which disconnects it in the weak sense.*

THEOREM D. *If M is a continuous curve which contains a simple closed curve k then k does not contain an uncountable set of points each of which disconnects M in the weak sense.*

Theorem C is a logical consequence of Theorems A and C.

Let D′ denote the theorem obtained from Theorem D by the substitution of the word *strong* instead of the word *weak*. I have recently established the following theorems,[§] of which the first includes Theorem D′ as a special case but is much more general than Theorem D′.

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[*] Loc. cit. Concerning the close relationship between this result and a result announced sometime before by Janiszewski see page 340.

[†] J. R. Kline has shown that if a continuum has no non-cut points it is connected im kleinen. Cf. this BULLETIN, vol. 27 (1921), p. 399. That such a continuum is unbounded follows from Theorem C.

[‡] Loc. cit.

[§] For proofs of the first three of these theorems see my paper, *On the*

THEOREM $\overline{\text{D}}$. *No closed and connected point set M (whether it be a continuous curve or not) contains a closed and connected subset K (whether K be a simple closed curve or not) which contains an uncountable set of points each of which disconnects* M but not K.*

THEOREM E. *In order that a bounded continuum M should be a continuous curve which contains no simple closed curve, it is necessary and sufficient that every sub-continuum of M should contain uncountably many points each of which disconnects M in the strong sense.*

THEOREM F. *In order that the continuous curve M should contain no simple closed curve it is necessary and sufficient that if K denotes the set of all those points of M that are not cut points of M, then no subset of K disconnects† M, even in the weak sense.*

THEOREM G.‡ *In order that a bounded continuum M should be a simple closed curve, it is necessary and sufficient that it should be disconnected by the omission of any two of its points.*

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cut points of continuous curves and of other closed and connected point sets, PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 9 (1923), pp. 101–106.

* That this theorem does not remain true if the word *disconnects* is replaced by the phrase *disconnects in the weak sense* may be seen if one considers the example of a point set M which is composed of a circle K together with a suitably selected simple spiral which has every point of K as a limit point. In this example, no point of K disconnects K either in the strong or in the weak sense but every point of K disconnects M in the weak (but not in the strong) sense. This example also shows that Mazurkiewicz' Theorem D would not remain true if the phrase *continuous curve* were replaced by the phrase *bounded, closed, and connected point set.*

† If N is a subset of M and $M - N$ is not connected then N is said to disconnect M. If $M - N$ is not strongly connected then N is said to disconnect M in the weak sense.

‡ Cf. my paper *Concerning simple continuous curves,* loc. cit. In a paper presented to the Society but, as far as I know, not as yet published, J. R. Kline has shown that in this theorem the stipulation that the set M is bounded may be replaced by the stipulation that M has no cut point. He also characterizes a simple closed curve as a continuum which is not disconnected by the omission of any proper connected subset. Cf. this BULLETIN, vol. 27 (1921), p. 399, and vol. 28 (1922), p. 8.

6. *Concluding Remarks.* Beyond certain propositions which hold true for both two and three dimensions, comparatively little is known concerning continuous curves in three-dimensional space. Various results obtained by Schoenflies, Miss Torhorst and others concerning the relation of a continuous curve to its complementary domains in the plane do not hold true in three-space.* For instance Schoenflies has shown that if a bounded plane continuum divides its plane into just two domains S_1 and S_2 such that every point of M is accessible from both S_1 and S_2 then M is a simple closed curve. In three-dimensional space however there exists a bounded continuum M which divides space into just two domains S_1 and S_2 such that (*a*) every point of M is accessible from both S_1 and S_2 and (*b*) the domain S_1 is uniformly connected im kleinen, but M is neither a simple closed surface nor any sort of continuous curve. I have shown,† that such a point set M would be a continuous curve if *both* S_1 and S_2 were uniformly connected im kleinen. It seems to me conceivable that some results concerning continuous curves and their complementary domains may be modified that they will hold true for three-space if, in addition to other changes, one replaces in certain connections the diameter of a point set by what I have called the two-dimensional extent of a point set.

Definition. A point set M is said to have a two-dimensional extent greater than e if there exists a right circular cylinder α with bases of diameter greater than e such that (*a*) neither base of α contains a point of M, (*b*) M contains at least one point in common with every simple continuous arc which has one end-point on one base of α, and its other end-point on the other base of α and lies, except for its end-points, wholly within α.

Every simple closed curve in a space S_2 of two dimensions can be thrown into any other simple closed curve in S_2 by a continuous one to one reciprocal transformation of S_2 into itself.* In other words in S_2 every two simple closed curves are *equivalent to each other*, in the strong sense, from the viewpoint of analysis situs. This proposition does not hold true for a space S_3 of three dimensions, as may be easily seen if one considers two simple closed curves in S_3 of which one is knotted and the other one is not. A knotted simple closed curve may be defined, from the viewpoint of analysis situs, as a simple closed curve (see definitions above) which is not a part of any simple closed † surface. It seems likely (*a*) that there exists no knotted simple continuous arc, that is to say, no simple continuous arc which does not lie on some simple closed surface, and (*b*) that any simple continuous arc in S_3 can be transformed into any other simple continuous arc in S_3 by a continuous one to one reciprocal transformation of S_3 into itself. But, as far as I know, no one has ever proved this. Indeed, as far as I know, there is nowhere in the literature any indication that anyone has *even considered* the question whether, if AB is an arc in a three-space S, and X is a point of S which does not belong to AB, then there exists an arc XB which has in common with AB only the point B. In other words, is an arc accessible at each of its ends from every point which does not lie on it?

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* See my paper, *On the relation of a continuous curve to its complementary domains in space of three dimensions*, loc. cit.

† Loc. cit.

A fruitful field for future investigation is afforded in that branch of analysis situs which has to do with various special types of continuous curves such as simple continuous arcs, simple closed curves and simple closed surfaces, in space of three dimensions. Little is known concerning the relation of such point sets to their complementary domains.

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* Cf. A. Schoenflies, MATHEMATISCHE ANNALEN, vol. 62 (1906), p. 324. Also J. R. Kline, *A new proof of a theorem due to Schoenflies*, PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 6 (1920), pp. 529–531.

† By a simple closed surface is meant a point set which is in one to one continuous correspondence with the surface of a sphere. For a topological characterization of such a point set, see the abstract of an unpublished paper by J. R. Kline and the author, this BULLETIN, vol. 28 (1922), p. 380. As far as I know, there does not exist in the literature a proof that a simple closed surface is the image of a sphere under a one to one reciprocal continuous transformation of S into itself. See, however, an abstract by J. W. Alexander, this BULLETIN, vol. 28 (1922), p. 10.

A GENERALIZATION OF A PROPERTY OF AN ACNODAL CUBIC CURVE

BY HAROLD HILTON

1. *Introduction.* We refer to the following property.

If in a plane cubic curve there is inscribed a real triangle ABC such that BC, CA, AB touch the curve at C, A, B, then the cubic can be projected by a real projection so as to have trigonal symmetry, i.e., it can be brought to self-coincidence by rotating it through $2\pi/3$ *about a point. If, in particular, the cubic is unicursal* (*rational*), *it must be acnodal.**

The generalization suggested is that of any unicursal curve in which a triangle ABC is inscribed, so that A, B, C are each given by a single value of a parameter in terms of which the coordinates of any point of the curve are rationally expressed, while the intersections of BC, CA, AB with the curve lie respectively p at C and q at B, p at A and q at C, p at B and q at A. We shall investigate the properties of such curves.

Take the parameters of A, B, C as 0, ∞, 1.† Then choosing suitable homogeneous coordinates, we have evidently

$$x : y : z = (t-1)^p : (-t)^p(t-1)^q : (-t)^q. \tag{1}$$

We shall find it convenient to use a quantity ϵ defined by

$$\epsilon \equiv p^2 - pq + q^2. \tag{2}$$

Elimination of t from (1) gives

$$x^{p/\epsilon}y^{q/\epsilon} + y^{p/\epsilon}z^{q/\epsilon} + z^{p/\epsilon}x^{q/\epsilon} = 0. \tag{3}$$

Hence the curves may be projected by a real projection so as to have trigonal symmetry, as in the case of the cubic. Points with parameters t, $1/(1-t)$, $(t-1)/t$ are those related by the symmetry. If $p = q$, p and q are factors of ϵ, and the curve is one of the "triangular-symmetric" curves discussed elsewhere.‡ We shall therefore suppose p and q unequal in what follows, and there is no loss of generality in supposing $p > q$. The methods of page 138 of H. P. A. C. show that the curve has no cusps other than those involved in the singularities at A, B, C; and that the only inflexions other than those at A, B, C are the three real inflexions given by

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* For each non-singular or acnodal cubic, two such real triangles exist.

† See Hilton, *Plane Algebraic Curves*, Clarendon Press, p. 148. This book is referred to later as "H. P. A. C."

‡ MESSENGER OF MATHEMATICS, vol. 50, (1921), p. 171.

$$pq(p-q)(t^3-3t+1) = \{(p^2-pq+q^2)(p+q-1) - 3pq^2\}(t^2-t). \tag{4}$$

The tangent at the point (1) is

$$t^p\{-pt+(p-q)\}x + (-1)^p(t-1)^{p-q}\{(p-q)t + q\}y + (-1)^q t^{p-q}(t-1)^p(qt-p)z = 0. \tag{5}$$

From each of the points A, B, C one real tangent can be drawn other than the sides of the triangle ABC. The parameters of the points of contact are $(p-q)/q$, $q/(q-p)$, p/q.

2. *Nodes.* Consider now the parameters t_1, t_2 of a node of the curve other than A, B, C. We have

$$\left(\frac{t_1-1}{t_2-1}\right)^p = \left(\frac{t_1}{t_2}\right)^p \left(\frac{t_1-1}{t_2-1}\right)^q = \left(\frac{t_1}{t_2}\right)^q. \tag{6}$$

Eliminating first $(t_1-1)/(t_2-1)$ and then t_1/t_2 from (6), we find that both these quantities are roots of the algebraic equation $x^\epsilon = 1$. We write therefore

$$t_2 = \omega t_1, \quad (t_2-1) = \Omega(t_1-1), \quad \text{where} \quad \omega^\epsilon = \Omega^\epsilon = 1. \tag{7}$$

Then (6) and (7) combined give

$$(1-\Omega)/(\omega-\Omega) \quad \text{and} \quad \varphi(1-\Omega)/(\omega-\Omega) \tag{8}$$

for the parameters of the node, subject to

$$\omega^\epsilon = \Omega^\epsilon = 1, \qquad \Omega^p = \omega^q, \qquad \Omega^q\omega^{p-q} = 1. \tag{9}$$

We now proceed to find the solutions of (9).

Let d be the highest common factor of p and q, and put

$$p = dp', \qquad q = dq', \tag{10}$$

where p', q' are prime to one another. We have

$$\omega = e^{2k\pi i/\epsilon}, \qquad \Omega = e^{2K\pi i/\epsilon}, \tag{11}$$

where k, K are any unequal integers included among 1, 2, 3, $\cdots$, $\epsilon - 1$, consistent with (9). Now (9) gives

$$\begin{cases} kq' - Kp' \equiv 0 \pmod{\epsilon/d}, \\ k(p'-q') + Kq' \equiv 0 \pmod{\epsilon/d}. \end{cases} \tag{12}$$

Elimination first of k and then of K from (12) shows that k and K are multiples of d. Put

$$k = dk', \qquad K = dK'. \tag{13}$$

Then the integers k', K' take all possible values $1, 2, 3, \cdots, \epsilon/d - 1$, consistent with

$$k'q' - K'p' \equiv 0 \pmod{\epsilon/d^2} \tag{14}$$

excluding the cases in which k' and K' are equal and multiples of ϵ/d^2. Each value of k' which is not a multiple of ϵ/d^2 gives d values of K', and each value of k' which is a multiple of ϵ/d^2 gives $d-2$ values of K'. Hence the number of possible values of ω and Ω is

$$d^2(\epsilon/d^2 - 1) + (d-2)(d-1) = \epsilon - 3d + 2.$$

Now the replacing of ω and Ω by $1/\omega$ and $1/\Omega$, respectively, merely interchanges t_1 and t_2. Hence the number of nodes other than A, B, C is $\frac{1}{2}(\epsilon - 3d + 2)$. If we replace ω, Ω by $1/\Omega$, ω/Ω we get a node obtained by the symmetry of the curve from that given by (7).

The coordinates of the nodes are got on replacing t in (1) by $e^{-k\pi i/\epsilon} \sin K\pi\epsilon \csc[(K-k)\pi/\epsilon]$, and $t-1$ by $e^{-k\pi i/\epsilon} \sin k\pi\epsilon \times \csc[(K-k)\pi/\epsilon]$, where k and K are multiples of d, such that

$$kq/d^2 - Kp/d^2 \equiv 0 \pmod{\epsilon/d^2};$$

d being the highest common factor of p and q, and ϵ being $p^2 - pq + q^2$, as stated earlier. We readily see that the value of t is complex and that the ratios of the coordinates are real. Hence the point-singularities of the curve, excluding A, B, C, are $\frac{1}{2}(\epsilon - 3d + 2)$ real acnodes.

3. *The Rationalized Equation.* The equations § 1 (1) give the degree n as $p + q$, and (5) gives the class m as $2p - q + 1$ $(p > q)$. Moreover, the curve is unicursal. Therefore Plücker's equations give, with the notation of page 112 of H. P. A. C.,

$$\begin{aligned} &n = p + q, && m = 2p - q + 1,\\ (15)\quad &\delta = \tfrac{1}{2}(p^2 + 2pq + q^2 - 3p - 9q + 8), && \kappa = 3(q-1),\\ &T = \tfrac{1}{2}(4p^2 - 4pq + q^2 - 8p + 7q), && \iota = 3(p - q). \end{aligned}$$

If we require the equation of the curve in rational form, we may proceed as follows. First suppose $d = 1$, so that p is prime to q. Then from (3) the rationalized equation is

$$\prod(\Omega^p x^{p/\epsilon} y^{q/\epsilon} + y^{p/\epsilon} z^{q/\epsilon} + \Omega^q z^{p/\epsilon} x^{q/\epsilon}) = 0,$$

where Ω is any root of $x^\epsilon = 1$. The equation is therefore obtained by equating to zero the circulant determinant, whose first row has $y^{p/\epsilon} z^{q/\epsilon}$, $z^{p/\epsilon} x^{q/\epsilon}$, $x^{p/\epsilon} y^{q/\epsilon}$ as first, $(q + 1)$th, $(p + 1)$th elements, the other elements being zero.

By Newton's diagram, or otherwise, we see that the singularity at A, B, or C is of the same nature as the origin in the curve with Cartesian equation

$$(16) \qquad 0 = (y^q - x^p) + \sum h x^\alpha y^\beta,$$

h being a constant, and α, β zero or positive integers such that $\alpha q + \beta p > pq$. The expansion of y in terms of x near the origin is of the form (see H. P. A. C., Chap. VI, § 1)

$$(17) \qquad y = x^{p/q}(a + b x^{1/q} + c x^{2/q} + \cdots).$$

Now suppose $d \neq 1$. First obtain the rationalized equation $f(x, y, z) = 0$ of the curve in which p and q are replaced by p' and q'. Then the required equation is

$$(18) \qquad \prod f(\omega_1 x^{1/d}, \omega_2 y^{1/d}, z^{1/d}) = 0,$$

where ω_1 and ω_2 are any roots of $x^d = 1$.

It is readily seen that the singularity at A, B, C is of the same nature as the origin in

$$(19) \qquad 0 = (y^{q'} - x^{p'})^d + \sum h x^\alpha y^\beta,$$

while the expansion of y near the origin is still of the form given in (17). The method of page 84 of H. P. A. C. shows that the polar reciprocal of the singularity is the singularity we get, if we replace q by $p - q$.

By § 2, the point-singularities other than A, B, C are merely $\frac{1}{2}(\epsilon - 3d + 2)$ acnodes. Hence, by (15), the singularities at A, B, C are each equivalent, so far as Plücker's equations are concerned, to $\frac{1}{2}(pq - p - 3q + d + 2)$ nodes and $(q - 1)$ cusps, and therefore to $\frac{1}{2}(p^2 - pq - 4p + 3q + d + 2)$ bitangents and $(p - q - 1)$ inflexions. There are besides $\frac{1}{2}(\epsilon + 4p - 2q - 3d - 6)$ other bitangents and 3 other inflexions. This statement may be verified by transforming (19) repeatedly by the substitution of xy for y or x, to analyze the singularity at the origin by the method of page 134 of H. P. A. C.

If we transform (3) by the quadratic transformation which consists in replacing x, y, z by $1/x$, $1/y$, $1/z$, we get a curve of the same type, but with $p - q$ replacing q. We may thus derive properties of one curve from those of the other, or duplicate properties of the same curve, if $p = 2q$. For instance, we see that there are exactly three conics through A, B, C which osculate the curve at a point other than A, B, C.

When the curve is projected so as to have trigonal symmetry about a point O, the circular points I, J lie on the curve unless $p + q$ is a multiple of 3. The parameters of I and J are $-\omega$ and $-\omega^2$, where $\omega^3 = 1$. If $p + q - 1$ is a multiple of 3, IJ is the tangent at I and J. If $p + q - 2$ is a multiple of 3, OI and OJ are the tangents at I and J. If $p + q$ is a multiple of 3, $t = -\omega$ and $-\omega^2$ give the point O, which is one of the acnodes. The general shape of the curve may readily be found by tracing roughly the curve with Cartesian parametral equations $x = (t-1)^p/t^q$, $y = t^{p-q}(t-1)^q$, into which it may be projected. There are four fundamentally different types, according as p and q are odd or even.

4. *Special Cases.* Some special cases may be noted.

If $p = 2$ and $q = 1$, $p = 3$ and $q = 1$, or $p = q$, the curve can be projected to have the symmetry of the equilateral triangle. Otherwise it can only have trigonal symmetry. In the first two cases there is a second triangle bearing the same relation to the curve as ABC.

If $p = 2$, $q = 1$, there is one acnode at $(1, 1, 1)$, and three collinear inflexions given by $t = 2 \cos 2\pi/9$, $2 \cos 8\pi/9$, $2 \cos 14\pi/9$. The points of contact of tangents from B and C, other than sides of the triangle ABC, are collinear with A, while A, B, C are "coincidence points." The points of contact of conics through A, B, C osculating the curve lie on another conic through A, B, C. Similar results hold for a non-singular cubic (H. P. A. C., p. 260, Ex. 15); but they do not hold for other values of p and q.

If $p = 3$, $q = 1$, there are three acnodes A', B', C'. Remembering the symmetry of the curve, we see that, if $A'B'C'$ is taken as triangle of reference, the equation of the curve can be put into the form

$$y^2z^2 + z^2x^2 + x^2y^2 + 2fxyz(x + y + z) = 0.$$

To obtain the constant f, we notice that the conic through the inflexions and the conic touching the inflexional tangents (H. P. A. C., p. 275) have the property that triangles can be inscribed in the first and circumscribed to the second. This gives $f = -3/4$. The two conics are $(x^2 + y^2 + z^2) = 2(yz + zx + xy)$, $5(x^2 + y^2 + z^2) = 6(yz + zx + xy)$. The conic through the inflexions touches the sides of the triangle $A'B'C'$.

We can prove readily that, conversely, any real trinodal quartic, whose inflexions-conic touches the three lines joining the nodes, can be projected into

$$2(y^2z^2 + z^2x^2 + x^2y^2) = 3xyz(x + y + z).$$

We may also notice the case of the unicursal quintic with three ordinary cusps such that each cuspidal tangent meets the curve in another cusp, and the case of the unicursal sextic with three rhamphoid cusps. The three osculating conics through the cusps of the quintic touch at collinear points.

5. *Further Generalizations.* We may generalize our problem still further, and consider the unicursal curve in which a triangle ABC is inscribed, so that A, B, C are each given by a single value of the parameter, while the intersections of BC, CA, AB with the curve lie respectively a at C and $n - a$ at B, b at A and $n - b$ at C, c at B and $n - c$ at A.

The equations (1), (2), (3) of § 1 are replaced by

(20) $$x : y : z = (t - 1)^a : (-t)^b(t - 1)^{n-b} : (-t)^{n-c},$$

(21) $$\epsilon \equiv n^2 - n(a + b + c) + (bc + ca + ab),$$

(22) $$y^{a/\epsilon}z^{(n-a)/\epsilon} + z^{b/\epsilon}x^{(n-b)/\epsilon} + x^{c/\epsilon}y^{(n-c)/\epsilon} = 0.$$

It is still true that there are three inflexions other than A, B, C, and that from each of A, B, C a single tangent can be drawn other than the sides of the triangle ABC.

The parameters of the nodes other than A, B, C are given by § 2 (3), § 2 (4) being replaced by

$$\omega^\epsilon = \Omega^\epsilon = 1, \qquad \Omega^a = \omega^{n-c}, \qquad \Omega^{n-b}\omega^{b+c-n} = 1.$$

We must postpone a more detailed discussion.

BEDFORD COLLEGE, REGENTS PARK, LONDON

ON UNIFORM LIMITEDNESS OF SETS OF FUNCTIONAL OPERATIONS[*]

BY T. H. HILDEBRANDT

1. *Introduction.* F. Riesz[†] has pointed out that the method of proving the Hellinger-Toeplitz theorem is closely related to certain proofs of boundedness of sequences of integrals. Essentially these state that a sequence of limited linear operations on a given class, bounded for every member of the class, is uniformly limited. The Hellinger-Toeplitz theorem is a special case of the theorem that a double sequence of limited bilinear forms, if bounded for every member of the class, is uniformly limited. The question naturally arises whether a theorem of the second type can be derived from a theorem of the first type by an iterative process. This paper derives such a theorem of the first type, by extracting the essential feature of the linearity condition, and utilizing some of the fundamental ideas of E. H. Moore and H. L. Smith.[‡]

2. *The Foundations.* We shall assume a general class $\mathfrak{P}$ of elements p; also a class $\mathfrak{M}$ of functions μ on $\mathfrak{P}$ to $\mathfrak{A}$ the class of complex numbers. The class $\mathfrak{M}$ is assumed to be linear (L), i.e., for every a_1 and a_2 of $\mathfrak{A}$ and μ_1 and μ_2 of $\mathfrak{M}$, $a_1\mu_1 + a_2\mu_2$ also belongs to $\mathfrak{M}$. We assume further that it is possible to define in $\mathfrak{M}$ a distance function or norm N, which is on $\mathfrak{M}$ to the real positive part of $\mathfrak{A}$ and has the property (Δ): for every a_1 and a_2, and μ_1 and μ_2

$$N(a_1\mu_1 + a_2\mu_2) \leqq |a_1| N\mu_1 + |a_2| N\mu_2.\text{[§]}$$

In terms of this norm it is possible to define the notion

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[*] Part of a paper presented to Society, March 26, 1921.

[†] *Équations Linéaires à une Infinité d'Inconnues*, footnote, p. 83.

[‡] *General theory of limits*, AMERICAN JOURNAL, vol. 44 (1922), pp. 102 ff.

[§] This condition (Δ) is equivalent to the two conditions

$$N(\mu_1 + \mu_2) \leqq N\mu_1 + N\mu_2, \qquad Na\mu_1 = |a| N\mu_1$$

for every μ_1, μ_2 and a. For from $Na\mu \leqq |a| N\mu$ it follows for $a \neq 0$ that

$$Na\mu \leqq |a| N\mu = |a| N\left(\frac{a}{a}\mu\right) \leqq \frac{|a|}{|a|} Na\mu = Na\mu.$$

distance as $N(\mu_1 - \mu_2)$ as well as *limit*, viz.,

$$\lim_n N(\mu_n - \mu) = 0.$$

In so far as we do not postulate that $N\mu = 0$ if and only if $\mu = 0$ for every p, the limit of a sequence μ_n thus defined is not necessarily unique. If two functions μ_0 and μ_{00} are limits of the same sequence then obviously $N(\mu_0 - \mu_{00}) = 0$.

We shall call a *Cauchy sequence* a sequence $\{\mu_n\}$ such that for every $e > 0$ there exists an n_e such that if $n_1 \geqq n_e$ and $n_2 \geqq n_e$, then

$$N(\mu_{n_1} - \mu_{n_2}) \leqq e.$$

We shall assume that the class $\mathfrak{M}$ is *complete* (Cp) with respect to N, viz., that every Cauchy sequence from $\mathfrak{M}$ has a limit in $\mathfrak{M}$.*

Finally we use some of the ideas suggested by E. H. Moore and H. L. Smith. We assume another general class: $\mathfrak{Q}$ of elements q. On $\mathfrak{Q}$ we assume a binary relation $R : q_1Rq_2$, with the properties

(T) if q_1Rq_2 and q_2Rq_3 then q_1Rq_3;

(C) for every q_1 and q_2 there exists a q_3 such that q_3Rq_1 and q_3Rq_2.

Our foundation thus consists of the system:

$$\sum = (\mathfrak{A};\, \mathfrak{P};\, \mathfrak{M}^{\text{on } \mathfrak{P} \text{ to } \mathfrak{A},\, L};\;\; N^{\text{on } \mathfrak{M} \text{ to } \mathfrak{A} \geqq 0,\, \Delta};\;\; \mathfrak{M}^{\text{Cp}};\;\; \mathfrak{Q};\;\; R^{\text{on } \mathfrak{Q}\mathfrak{Q},\, TC}).$$

3. *The Fundamental Theorem.* Our theorem concerns a set of operations T_q on $\mathfrak{M}$ to $\mathfrak{A}$, which have the following properties:

(Δ = *triangle*), for every q, a_1, a_2, μ_1, μ_2

$$|T_q(a_1\mu_1 + a_2\mu_2)| \leqq |a_1T_q(\mu_1)| + |a_2T_q(\mu_2)|;$$

(M = *modular or limited*), there exists an M_q such that for every μ

$$|T_q(\mu)| \leqq M_qN\mu.$$

We define as the *modulus* M_q of T_q the least upper bound of the values $|T_q(\mu)|$ for $N\mu = 1$.

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* The germ of such a general class $\mathfrak{M}$ is suggested by Riesz, ACTA MATHEMATICA, vol. 41 (1918), p. 72. It has been used by Bennett, PROCEEDINGS OF THE NATIONAL ACADEMY, vol. 2 (1916), p. 595; Lamson, AMERICAN JOURNAL, vol. 42 (1920), p. 245; Hahn, MONATSHEFTE, vol. 32 (1922), pp. 4 ff.; Banach, FUNDAMENTA MATHEMATICÆ, vol. 3 (1922), pp. 135 ff.

The set T_q is *semi-uniformly bounded* over $\mathfrak{M}$,* i.e., there exists a sequence $\{q_m\}$ such that for every μ there exists an m_μ and an e_μ such that if qRq_{m_μ} then

$$| T_q(\mu) | \leqq e_\mu.$$

We note that on account of the properties T and C of R the sequence $\{q_m\}$ may always be assumed to be monotonic increasing, i.e., for every m $q_m R q_{m-1}$. For if the property is holding for a given sequence $\{q_m\}$ it will also hold for the sequence $\{q_m'\}$, where $q_1' = q_1$, and q_m' is such that $q_m' R q'_{m-1}$ and $q_m' R q_m$. Our theorem is as follows.

FUNDAMENTAL THEOREM. *A set of operations T_q on $\mathfrak{M}$ to $\mathfrak{A}$ having the properties Δ and M, which is semi-uniformly bounded over $\mathfrak{M}$, is ultimately uniformly limited, i.e., there exists a q_0 and an M such that for qRq_0*

$$M_q \leqq M.$$

This q_0 is a member of the sequence involved in the semi-uniform boundedness.

The proof of the theorem follows the lines of proofs of similar theorems in the literature.† Assume that the conclusion is not valid. Then the following statement would be true:

(A) For every $e > 0$ and every q_m of $\{q_m\}$ there exists a q_{me} such that $q_{me} R q_m$ and $M_{q_{me}} > e$.

Let $\{e_n\}$ be a sequence of numbers approaching infinity. Then by a step by step process we can determine c_n, $e^{(n)}$, m_n, μ_n, q_{en}, satisfying the following conditions:

(1) $c_1 = 1$, c_n is the least of the quantities

$$\frac{e_1}{2^n M_{q_{e1}}}, \quad \cdots, \quad \frac{e_{n-1}}{2^2 M_{q_{e,\, n-1}}};$$

(2) q_{en} and μ_n such that $N\mu_n = 1$, $q_{en} R q_{m_{n-1}}$ and

$$| T_{q_{en}}(\mu_n) | > \frac{e_n + e^{(n-1)}}{c_n};$$

(3) m_n and $e^{(n)}$ such that qRq_{m_n} implies $| T_q(\Sigma_1^n c_i\mu_i) | < e^{(n)}$.

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* Cf. Moore and Smith, loc. cit., p. 114.

† Cf., for instance, Lebesgue, ANNALES DE TOULOUSE, (3), vol. 1 (1909), p. 61; Hahn, loc. cit., p. 9; Banach, loc. cit., p. 157.

The first of these is obviously possible. It insures the fact that $\Sigma_1^\infty c_n$ is convergent and

$$\Sigma_m^\infty c_n < \frac{e_{m-1}}{2M_{q_{e,m-1}}} \cdot$$

For (2) we apply statement (A) to $q_{m_{n-1}}$ and $(e_n + e^{(n-1)})/c_n$. The condition (3) follows from the semi-uniform boundedness of the set T_q. From (2) and (3) and from the property Δ of T_q it follows that

$$\begin{aligned} |T_{q_{en}}(\Sigma_1^n c_i\mu_i)| &\geqq \Big|\, |T_{q_{en}}(c_n\mu_n)| - |T_{q_{en}}(\Sigma_1^{n-1} c_i\mu_i)| \,\Big| \\ &> c_n \frac{e_n + e^{(n-1)}}{c_n} - e^{(n-1)} \\ &= e_n. \end{aligned}$$

Now since the series $\Sigma_1^\infty c_n$ is convergent, it follows that the sequence $\Sigma_1^m c_n\mu_n$ is a Cauchy sequence. For

$$N(\Sigma_k^m c_n\mu_n) \leqq \Sigma_k^m c_n N\mu_n = \Sigma_k^m c_n \to 0.$$

Hence there exists a μ of the class $\mathfrak{M}$ such that

$$\lim_m N(\mu - \Sigma_1^m c_n\mu_n) = 0.$$

Then by the Δ property of N we can show that

$$N(\mu - \Sigma_1^m c_n\mu_n) \leqq \Sigma_{m+1}^\infty c_n.$$

Hence

$$\begin{aligned} |T_{q_{en}}(\mu)| &= |T_{q_{en}}(\mu - \Sigma_1^n c_i\mu_i + \Sigma_1^n c_i\mu_i)| \\ &\geqq \Big|\, |T_{q_{en}}(\Sigma_1^n c_i\mu_i)| - |T_{q_{en}}(\mu - \Sigma_1^n c_i\mu_i)| \,\Big| \\ &\geqq e_n - M_{q_{en}} N(\mu - \Sigma_1^n c_i\mu_i) \geqq e_n - M_{q_{en}}\Sigma_{n+1}^\infty c_i \\ &\geqq e_n - e_n/2 = e_n/2, \end{aligned}$$

i.e., since e_n approaches infinity with n, $|T_{q_{en}}(\mu)|$ approaches infinity with n. But from the semi-uniform boundedness of T_q, it follows that there exists an e_μ and an m_μ such that for qRq_{m_μ} it is true that

$$|T_q(\mu)| < e_\mu.$$

Since the sequence q_m is monotonic increasing and $q_{en}Rq_{m_{n-1}}$, it would follow that the sequence $|T_{q_{en}}(\mu)|$ is bounded. We have thus reached a contradiction, showing that statement (A) is untenable. Hence the theorem is proved.

We can obtain various special cases of this theorem by replacing the hypotheses by special cases. If we assume, for instance, that $T_q(\mu)$ converges semi-uniformly over $\mathfrak{M}$,* then the set $T_q(\mu)$ is semi-uniformly bounded over $\mathfrak{M}$. Hence we have the following corollary.

COROLLARY I. *If T_q is a set of operations on $\mathfrak{M}$ to $\mathfrak{A}$ which have the properties Δ and M, for every q, and*

$$\lim_q T_q = T \ (\mathfrak{M};\ \textit{semi-uniformly}),$$

then there exists a q_0 and an M such that for qRq_0 it is true that

$$M_q \leqq M,$$

where M_q is the modulus of T_q. As a consequence T also has the properties Δ and M with modulus less than or equal to M.

If T is linear, i.e., if for every a_1 and a_2, and μ_1 and μ_2

$$T(a_1\mu_1 + a_2\mu_2) = a_1 T(\mu_1) + a_2 T(\mu_2),$$

then T has the property Δ, so that we have the following corollary.

COROLLARY II. *If T_q is a set of linear limited operations on $\mathfrak{M}$ to $\mathfrak{A}$, which is semi-uniformly convergent over $\mathfrak{M}$ to T, then the set T_q is ultimately uniformly limited and T is a limited linear operation on $\mathfrak{M}$ to $\mathfrak{A}$.*

Further specialization is possible by taking a particular class $\mathfrak{Q}$. For instance suppose $\mathfrak{Q}$ is the class of positive integers. Then the set T_q becomes a sequence of operations T_n and the semi-uniform boundedness or convergence becomes simply boundedness or convergence, respectively, for every μ. In this form the theorem includes a wide variety of cases, depending on the nature of the class $\mathfrak{M}$ and the norm N.†

We obtain another corollary of this theorem by assuming that the T_q are transformations on $\mathfrak{M}'$ to $\mathfrak{M}''$, where $\mathfrak{M}'$ and $\mathfrak{M}''$ are on $\mathfrak{P}'$ and $\mathfrak{P}''$ respectively to $\mathfrak{A}$, and have the same properties as the class $\mathfrak{M}$, i.e., linearity, the existence of a norm with property Δ relative to which the classes are complete. It is possible to define the concepts limited or modular, and semi-uniform boundedness over $\mathfrak{M}'$ for a set T_q by replacing in the definition for operations on $\mathfrak{M}$ to $\mathfrak{A}$ the absolute value signs by the norm N''. Then we have the following corollary.

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* Cf. Moore and Smith, loc. cit., p. 114.

† Hahn (loc. cit.) gives a large list of special classes and norms. The fact that a linear form, if convergent for every point of Hilbert space, is limited, is obviously also a special case of this theorem.

COROLLARY III. *If a set of linear limited transformations* T_q *on* $\mathfrak{M}'$ *to* $\mathfrak{M}''$ *is semi-uniformly bounded as to* $\mathfrak{M}'$, *then the set is ultimately uniformly limited, i.e., there exists a* q_0 *and an* M *such that* $M_q \leqq M$ *or* $N''T_q(\mu') \leqq MN'\mu'$ *for every* qRq_0 *and* μ'.*

4. *Bilinear Operations.* An operation B on $\mathfrak{M}'\mathfrak{M}''$ to $\mathfrak{A}$, where $\mathfrak{M}'$ and $\mathfrak{M}''$ have the properties of $\mathfrak{M}$, is *bilinear* if for every μ_1', μ_2', μ_1'', μ_2'', a_1 and a_2 it is true that

$$B(\mu_1' + \mu_2', \mu_1'' + \mu_2'') = B(\mu_1', \mu_1'') + B(\mu_2', \mu_1'') + B(\mu_1', \mu_2'') + B(\mu_2', \mu_2''),$$

and

$$B(a_1\mu_1', a_2\mu_1'') = a_1a_2B(\mu_1', \mu_1'').$$

It is *limited* or *modular*, if there exists a constant M such that for every μ' and μ''

$$|B(\mu', \mu'')| \leqq MN'\mu'N''\mu''.$$

M is called the *modulus* of B when it is the least upper bound of $|B(\mu', \mu'')|$ for all $N'\mu' = 1$ and $N''\mu'' = 1$. Then we have the following theorem.

THEOREM. *A set of bilinear limited operations* B_q *on* $\mathfrak{M}'\mathfrak{M}''$ *to* $\mathfrak{A}$ *which is semi-uniformly bounded over* $\mathfrak{M}'\mathfrak{M}''$ *is ultimately uniformly limited.*

Let $T_q(\mu')$ be the least upper bound of $|B_q(\mu', \mu'')|$ for $N''\mu'' = 1$. This will exist for every q and μ' and will have the following properties:

$$(\Delta) \qquad T_q(a_1\mu_1' + a_2\mu_2') \leqq |a_1T_q(\mu_1')| + |a_2T_q(\mu_2')|,$$

$$(M) \qquad T_q(\mu') \leqq M_qN'\mu',$$

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* By specializing $\mathfrak{Q}$, $\mathfrak{M}'$ and $\mathfrak{M}''$, we get a set of interesting theorems, including for instance some of interest in the definition of the summability for a series or integral. Cf., for instance, Kojima, TÔHOKU JOURNAL, vol. 12 (1917), pp. 291 ff.; Carmichael, this BULLETIN, vol. 25 (1918), p. 118; Schur, JOURNAL FÜR MATHEMATIK, vol. 151 (1920), pp. 82 ff.

the set $T_q(\mu')$ is semi-uniformly bounded. For if we consider $B_q(\mu', \mu'')$ as a function of μ'' keeping μ' fixed, we can apply the fundamental theorem. If $\{q_m\}$ is the sequence involved in the semi-uniform boundedness of B_q then for every μ', there will exist an $M_{\mu'}$ and an $m_{\mu'}$ such that for $qRq_{m_{\mu'}}$ and every μ''

$$|B_q(\mu', \mu'')| \leqq M_{\mu'} N''\mu'' \quad \text{or} \quad T_q(\mu') \leqq M_{\mu'}.$$

It follows that $T_q(\mu')$ has the properties involved in the hypothesis of our fundamental theorem, and hence there exists an m_0 and an M such that for qRq_{m_0} and every μ' it is true that

$$T_q(\mu') \leqq MN'\mu'.$$

Consequently for qRq_{m_0} it is true that

$$|B_q(\mu', \mu'')| \leqq T_q(\mu')N''\mu'' \leqq MN'\mu'N''\mu''$$

which is the ultimate uniform limitedness desired.

We can make specializations, as in the case of the fundamental theorem. The Hellinger-Toeplitz theorem is a special case of this theorem if $\mathfrak{M}' = \mathfrak{M}''$ = Hilbert space, $\mathfrak{Q}$ is the class of all pairs of positive integers, and

$$B_q(\mu', \mu'') = \Sigma_1^n \Sigma_1^m \beta_{ij} \mu_i' \mu_j'',$$

where β_{ij} are the coefficients of a bilinear form convergent for every μ', μ'' of Hilbert space. Obviously the sequence q_m of the semi-uniform boundedness can be taken to be the set (m, m) for all m.

Apparently, we can extend this theorem to that of a set of bilinear transformations on $\mathfrak{M}'\mathfrak{M}''$ to $\mathfrak{M}'''$, or to a set of multilinear operations or transformations. From the latter it would be possible to derive the extension of the Hellinger-Toeplitz theorem to multilinear forms.

THE UNIVERSITY OF MICHIGAN

BECK ON COORDINATE GEOMETRY

Koordinaten Geometrie. Volume 1, *Die Ebene.* By Hans Beck. Berlin, Julius Springer, 1919. x + 432 pp.

A celebrated New York divine of an earlier generation was once asked if he believed that he had a soul. He replied: "I do not understand the bearing of the question. I *am* a soul, I *have* a body."

Exactly the same distinction of verbs appears in the study of geometry. For historic or didactic reasons the student begins with synthetic geometry. The point is an undefined datum, about which more or less explicit assumptions are made. The geometry of points, lines, circles, etc., is carried to a considerable distance without any serious analytic help. After this analytic geometry is begun, and the student learns that, in order to present his points in well-bred society, he must clothe each with two numbers, called "coordinates." A moment's reflection shows him, however, that this is another case where "the apparel proclaims the man." What counts is not the undefined point, but the definite coordinates, the point is a sort of useless Kantian "thing-in-itself" that might as well be discarded.

The first writer to set this in perfectly clear light was Klein in his *Erlanger Program.* Here are various manageable groups of quantities, independent quantities, homogeneous quantities, quantities connected by certain relations. Here, further, are certain groups of transformations, which operate on these quantities; here are invariants and covariants of the groups. There are all sorts of conceivable sets of names to attach to the quantities, groups, invariants, etc. It is the path of wisdom to choose names which fit well into our geometric preconceptions and forms of speech. and to interpret in terms of them the results of our analytic manipulations, but we are at liberty at any moment to throw our vocabulary overboard and introduce new terms. Two geometries are equivalent if built on the same transformations and invariants, regardless of the shape of the letters used, or the names attached to them. At the very outset we do not say that a point *has* two coordinates, but that it *is* two coordinates. When the number field for the coordinates is real, this form of speech may seem a little artificial, but it is certainly legitimate. When, however, the number field is complex, a so-called "geometric" equivalent corresponding to the complex coordinates is bound to be more or less artificial, and the complete identification of point and coordinates by defining the former as nothing more nor less than the latter is not only natural but almost inevitable.

The present work, which takes this point of view from the outset, epitomizes and completes the results reached by a number of geometers during the last half century, the greatest of whom is Study. The fact is that the discerning reader will continually perceive the figure of Study behind a large part of what Beck has written to the extent that he may even feel inclined to exclaim:

"The voice is Jacob's voice, but the hands are the hands of Esau."

This means, of course, that the detail is very carefully done; let us look at it somewhat more closely.

In Chapter I a point is defined as a pair of complex coordinates, a line as the totality of points whose coordinates satisfy a linear equation. A sharp distinction is drawn between an isotropic and a non-isotropic line; through each point will pass just two lines of the former category. If a non-isotropic line l be given, we obtain an involutory collineation by exchanging each point of the plane with that other point whose two isotropics meet l in the same two places. This is called a *reflection* in the line l. The distance of two points is defined, analytically of course, and it is shown that distances are invariant under such reflections. Translations are obtained as the products of reflections in pairs of parallel lines, rotations as the products of reflections in intersecting lines; the whole euclidean group for the plane is built in a simple and satisfactory manner.

In Chapter II the center of interest is the straight line. The line has three homogeneous coordinates, and the idea of homogeneity is thoroughly discussed, although, curiously enough, homogeneous point-coordinates are not introduced until thirty pages later. The tangent of the angle of two lines is defined by the usual rational formula, but the tangent of the angle is not the function we want, since its period is π; we want a function with the period 2π, namely, the cosine; and this, unfortunately, is not rationally expressible in terms of the coordinates of the line. To avoid the difficulty, resort is had to the familiar process of orienting the line. Every set of numbers, not all zero, proportional to four quantities $u_0 : u_1 : u_2 : u_3$, where $-u_0^2 + u_1^2 + u_2^2 + u_3^2 = 0$, shall be called an *oriented line* or a *spear* (German, *Speer*). This spear is said to be *on* the line $u_1x + u_2y + u_3 = 0$. The cosine of the angle of two spears is given by the formula

$$\cos(u, v) = \frac{u_1v_1 + u_2v_2}{u_0v_0}.$$

The distinction between a line and a spear is most carefully made, and it is pointed out how positive and negative distances may be rationally separated on a spear. The latter part of the chapter deals with vectors and *staffs*, i.e., vectors limited to a single line; it is a real pity that we have no good English name for these quantities except, perhaps, forces.

Chapter III is headed *Collineations*, and deals with the fundamental properties of these transformations, and with their classification. I must confess that at this point the treatment seems to me somewhat labored. It is a point of honor with the author to put every formula not only in absolutely accurate form, but in its most general form. He scorns such artifices as simplifying a problem by a change of coordinates. And when you come to think of it, he may not do this; for if you do not say that a point *has* two coordinates, but that it *is* two coordinates, why then you cannot change the coordinates and have the same point. He has such analytical skill that he suffers less from this puritanic point of view than would most writers, but he devotes fourteen very full pages to exhibiting an example of each of five general types of plane collineations, and then four more pages to the characteristic equation and the proof that all possible types have been covered. The chapter ends with sections discussing affine transformations, stretchings, and systems of circles, oriented and non-oriented.

The fourth chapter is headed *Correlations*, but this is a misnomer. The principal use made of correlations is to define a conic according to Steiner's scheme. The bulk of the chapter is given to those collineations, happily called *automorphic*, which leave a conic in place. These are turned over on every side, and expressed in various ways, including quaternion form. What the writer is preparing for is the Cayleyan projective-metric, and this is taken up with great care. The radicals in both distance and angle formulas are avoided by the startling device of orienting both points and lines. The non-euclidean trigonometry is developed very prettily, and the already familiar euclidean metrical formulas are reached by a limiting process. A curious use of words is that whereby Lobachevski parallels, lines intersecting on the absolute conic, are defined as *paratactic;* this term was first introduced by Study in 1902 * to denote Clifford parallels, skew lines at a constant distance in three-dimensional space.

The domain in the first four chapters is the complex one. Chapter V, the last, concerns itself with real figures, real points, real collineations, real conics, etc. Circle geometry in the Gauss plane appears near the close.

And what shall we say of the book as a whole? The sketchy account above can give but the faintest idea of the amount of material; for, besides the main text, there are endless pages of problems and suggestions in small type. As mentioned above, every one of the bewildering assortment of formulas is stated with minute care, developed with conscientious rigor, and exhibited in its most general form. This is no easy accomplishment, as every geometer knows. One has the impression of a compendium, where the last word is said in deducing a large number of formulas; or, to change the metaphor, a vast arsenal where many perfect tools are built and stored for the benefit of future attacks on the difficulties of geometry. The writer could not accomplish his result without paying the price. He paid it, and the reader must pay also. The book is never obscure, the manipulations are not over-difficult, but the total effect is rather overwhelming. There are countless formulas developed for no apparent reason; there are scarcely any theorems or applications. The method is always the same: careful, straightforward conscientious algebraic analysis, with skillful use of a small number of simple algebraic identities, much as in the classical invariant theory. It is all very perfect, also very fatiguing. One has the impression that the writer is somewhat lacking in suppleness and imagination. A reviewer once said of one of Salmon's books that the reader had the instinctive feeling that the writer always hit on just the right method to solve each problem. The late Gaston Darboux once told me that it seemed to him that the essence of geometry consisted in finding in each individual problem the best method for its solution. But Beck will none of all this. Rigorous formalism must be carried through to the end, though the heavens fall.

J. L. Coolidge

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* Jahresbericht der Vereinigung, vol. 11.

KEYNES ON PROBABILITY

A Treatise on Probability. By J. M. Keynes. London, Macmillan and Co., 1921. xi + 466 pp.

Starting as a Fellowship Dissertation, interrupted by the war, preceded by economic publications widely circulated, and now published, apparently without all the finishing touches the author had liked to bestow, Keynes's *Treatise on Probability* is (see p. 433) the first systematic work in the English language on the logical foundations of probability since Venn's *Logic of Chance* (1866). Carefully documented throughout, the work closes with a 25 page bibliography which, though not pretending to completeness, is a much longer list of what has been written about probability than can be found elsewhere (p. 431). As the author has made these statements himself, the reviewer has but to quote. Two other quotations may be of interest: (1) "The theory of 'Testimony,'—of the combination of the testimony of witnesses,—has occupied so considerable a space in the traditional treatment of Probability that it will be worth while to examine it briefly. It may, however, be safely said that the principal conclusions on the subject set out by Condorcet, Laplace, Poisson, Cournot and Boole are demonstrably false. The interest of the discussion is chiefly due to the memory of these distinguished failures" (p. 180). Never perhaps since ancient biblical times has such a redoubtable army of philistines been so deftly slain (*Judges*, XV, 15). And (2) in speaking of Poincaré's opinions on probability in his *Calcul des Probabilités* and *Science and Hypothesis*, "neither of which appears to be in all respects a considered work," * we find (p. 84): "He seems to endeavor to save his reputation as a philosopher by the surrender of probability as a valid conception, without at the same time forfeiting his claim as a mathematician to work out probable formulæ of practical importance." Apparently our author little realizes that, stumbling in the dark, he has hit upon a very clever characterization of Poincaré's real philosophic system—pragmatism. The Fellowship Dissertation stage still lingers in spots.

To understand Keynes one has constantly to bear in mind that he is of the philosophic not of the mathematical school, that he derives from Leibniz, Hume and Venn rather than from Bernoulli, Laplace, and Charlier. He is seeking to lay a logical foundation for probability and to examine that foundation in the light of a wide reading of philosophic, mathematical, and statistical works on probability. He is not uninfluenced by the

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* Keynes's request (p. 427) for generosity on the part of his readers is not so likely to be granted as if his own manners conformed to that spirit; besides, when writing a really critical treatise an author defeats his own ends unless he so writes that readers need not have to discount either his scientific exposition or his professional manner. If taken literally, Keynes's request must undermine our confidence in his own belief that he has accomplished the very thing that he set out to do!

Russell-Whitehead school of rigorous mathematics. Indeed, Part II, Fundamental Theorems, pp. 115–214, is much cast in their style. Much of this part is hard reading because so compact, but he would be rash indeed who would compare its difficulty with the *Principia Mathematica*; the resemblance is, fortunately for most of us, rather to the earlier *Principles of Mathematics*. Other parts are hard reading to one not used to philosophic discourse. For example (p. 36) the author is trying to be very explicit and very clear on a very important point: "A book bound in blue morocco is more like a book bound in red morocco than if it were bound in blue calf; and a book bound in red calf is more like the book in red morocco than if it were blue calf." Whether it is a deficiency in our visual or in our tactile sense or, even worse, in our ratiocinative powers which makes this statement, whether in or out of its context, unintelligible has been a distressing trilemma ever since we read it. Or is it the War or the Peace or only the Fellowship Dissertation again?

By this time not only the author but our readers will have perceived that we are not fit to review this book. There are others who appear to share our predicament. The keen mind of one of our ablest statisticians, Arne Fisher, rebels at Keynes's criticisms and shows its fire in the last section (pp. 277–279) of his *Mathematical Theory of Probabilities*, 2d edition, Macmillan, 1922. It would be interesting to quote Fisher at some length, for what an author will incorporate in a text often carries more weight than what a reviewer may write, but let the readers of the BULLETIN look up the reference and read the whole. Also let them not take too seriously what Mr. Fisher or I may say. Neither of us is primarily a philosopher, economist or logician; we have both been under the necessity, an enjoyable necessity, of using probability in a wide range of problems. The attitude toward a subject of those who use it is different from that of those who think on it. This phenomenon is well known in mathematics. The progress towards the greater use of a subject is made by those who use it, the progress towards its greater digestion and codification is made by those who think on it. Although the attitude and the method of work of these two camps are different, often so different as to be mutually incomprehensible, it is not necessary that they be mutually destructive.

Some of Keynes's comments on illustrious investigations and originators in probability might have been cast in far different language had he realized the essentially pragmatic, the unconsciously pragmatic, point of view of the scientific as contrasted with the philosophic investigator. I have recently carefully reread Laplace's *Essai philosophique sur les Probabilités*, and I cannot refrain from noting what appears to me a patient, critical, pragmatic, and tentative attitude on the part of Laplace. His treatment whether of the Law of Non-sufficient Reason, of the Law of Succession, or even of the previously mentioned "Testimony" seems open-minded and fair. But I can understand how it may seem very different to Keynes. A scientist writes a direct, clear style with definite statements; he does not qualify everything indefinitely even when he realizes that qualification would be necessary for generalization beyond the case in hand. He writes for his confrères, who understand his style. Take the case of many a physicist who has written on that particular statistical theorem known as the Law of the Equipartition of Energy, and who has not, because he has not thought it necessary, made the statement that the law cannot possibly be true * because of the values of the ratio of the specific heats. To the philosopher or economist who may rarely make a definite statement, and who from the nature of his work and his interests perhaps cannot, the definite style of the scientist appears dogmatic, rather than pragmatic, and altogether damnable—but it is written: Judge not, that ye be not judged.

I do not, however, wish to give the impression that Keynes's book is not worth reading and study, that it was not worth doing and on the whole well done. The reason that I have dealt at such length with certain characteristics which mar the work is because, writing for those who may easily be irritated by these blemishes, I wish to explain them in the hope of reducing the irritation to such a point that the readers of this review will persevere through them to the real enjoyment and instruction that I have found in many parts of the work. The analysis of the fundamental concepts connected with the category of probability (Part I) is in many respects illuminating.† The logical treatment, in the style of Russell, of the fundamental theorems is also appealing (Part II). The general advice to "watch your step" is perhaps much less necessary to the scientist than to the philosopher because the former is dealing, as Poincaré frankly says in language only partly understood by the author, with specific cases, and perhaps only tentatively, where external indicia of impending danger or safety are not lacking, whereas the latter is developing generalities that may all too readily entice him without warning into real trouble.

There is much to interest the statistician in the discussion of the different frequency distributions resulting therefrom (pp. 194–214). Highly valuable, too, is the discussion, so far as it goes, of the normal or Gaussian law and of the Tchebycheff criterion which is so much safer than the probable error in all cases where the observations are few or are not distributed according to the normal law (pp. 351–359). The analysis of the ideas of Lexis is good but could have been carried further to include Charlier and the whole Scandinavian school who have been very active in the last forty years and who seem to be neglected by Keynes, and not by him alone. To the long bibliography one might add the essay of Royce on *The mechanical, the historical and the statistical*, SCIENCE, April 17, 1914, pp. 551–566. There are points here of philosophic import that seem not to have been covered by Keynes and might be worked into his system. Moreover some might feel that the *Statistical Mechanics* of Gibbs (1901), when viewed philosophically, particularly with respect to the meaning of equilibrium and of natural systems (whether physical or economic) and collated with the very general viewpoint of Royce and of C. S. Peirce (whose maturer work Keynes does not cite), might be worthy of at least a bibliographic reference by an author who is setting up a category of probability. However, it would be unreasonable to expect any discussion of categories to reach nearer the date of issue than about 50 years, just as one can hardly expect the full treatment of the necessary and sufficient conditions justifying a new analytical method to follow right on the heels of the introduction of such a method by the physicist (Fourier, Heaviside).

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* See the comment of Gibbs in the preface to his *Statistical Mechanics*, especially p. x.

† Keynes seems to belong rather to the a priori than to the a posteriori school of thought. This is natural from his philosophical, logical, analytical viewpoint. To adopt the a posteriori attitude consistently in a didactic treatise is difficult and rare; Thiele tried it relentlessly and I think that not many read his work, that fewer find it easy to read.

E. B. Wilson

BLASCHKE ON DIFFERENTIAL GEOMETRY

Vorlesungen über Differentialgeometrie und geometrische Grundlagen von Einstein's Relativitätstheorie. Volume 1, *Elementare Differentialgeometrie.* By W. Blaschke. Berlin, Julius Springer, 1921. x + 230 pp.

This volume is the first of a series of three which the author plans to publish under the first part of the title given above. It is devoted in part to the classical theories of the differential geometry of curves and surfaces, and in part to some very interesting special chapters of these theories with which the author himself has been especially occupied. In his preface Professor Blaschke says that this first volume contains a presentation of the properties of curves and surfaces which are invariant under the group of motions, that the second will be devoted to affine differential geometry, and that a third will present the geometrical theories of Riemann and Weyl, which are so closely related to the Einstein theory of gravitation.

There are two interesting features of the book that attract one's attention at the very start. The first is the consistent use of vector notations. All of us who have lectured on differential geometry have doubtless been impressed with the great economies in presentation which these notations would afford. I have myself hesitated to use them in lecture courses because of the loss of time necessitated at the beginning of a course by explanations to hearers who have had no experience with vector notations, and because of the slight element of awe and mystification which these notations seem to arouse in the minds of those who have had only a limited acquaintance with them. After reading Professor Blaschke's book, I have grave doubts of the correctness of my attitude. The properties of vectors which one needs in differential geometry are few and simple, and he has demonstrated that they may be clearly and concisely explained as the occasions for their use arise. When one considers the many applications of vector analysis in other domains as well as in geometry, it seems clear that we should acquaint our students with the elements of the subject at the earliest possible moment. Mathematical physicists usually shy away from abstract mathematical notations, as they did from the non-euclidean theories of space before the recent revolution. Is it not curious that they should be the leading advocates of the vector analysis notations which mathematicians in equally appropriate situations have in many cases studiously avoided?

The second outstanding feature of the book is the fifth chapter, devoted to the macroscopic properties of surfaces as contrasted to the differential properties with which most of us are more familiar. The only simply closed analytic surfaces with constant Gaussian curvature are spheres. If we designate an analytic simply closed convex surface by the name oval surface, then the only oval surfaces with constant mean curvatures are spheres. On every oval surface there are at least three closed geodesics and the image of each of them on the Gaussian sphere of the surface bisects the area of the sphere. These are samples of the theorems which this chapter contains. At the present time comparatively few results in the macroscopic theory of surfaces have been attained, most of them being properties of oval surfaces. The proofs which lead to them, as Professor Blaschke says in his preface, are among the most difficult known in any branch of analysis. It seems to me that the author has presented these proofs on the whole with great clearness, and we may feel confident that in this domain, in which he himself has had great interest, a comprehensive summary of known results has been given.

Throughout the book one notices frequent applications of the theory of the calculus of variations. This is inevitable in one form or another in the treatment of geodesics and minimal surfaces, but the author is well acquainted with the theory and apparently welcomes opportunities to apply this knowledge. In no case are theorems used, however, which are not established in the text. One may look forward to still further interesting applications of the calculus of variations in the third volume promised by the author.

The book is written in concise and "snappy" style, but the sequences of logical steps are clear and the text always interesting. References to original sources and historical remarks are frequent. The latter are sometimes more than mere paragraphs. On pages 80–81 and 123, for example, are very interesting historical sketches devoted to Monge and Gauss. In a number of places in the book computations are left to the reader. These are sometimes of considerable extent, but as I read the book I found only a few instances in which additional suggestions from the author would have helped to a more rapid understanding. At the ends of chapters are lists of problems and of theorems to be proved by the reader. These are in many cases not elementary, but references are usually given. I should regard the lists as on the average unsuited to the needs of beginning students, but suggestive of much interesting study and reading outside the limits of the book itself for readers of all types.

Chapter I is devoted to vector notations and the differential geometry of twisted curves, with applications to Bertrand curves, "Böschungslinien," evolutes and involutes, and isotropic curves. Böschungslinien are curves whose tangents make a constant angle with a fixed direction in space, and the author determines all such on a sphere or a paraboloid of revolution. In § 9 a foretaste of the later macroscopic chapter is given by a proof of the theorem that every oval plane curve has at least four vertices, that is, four points where the rate of change of the curvature with respect to the length of arc is zero. One misses, in this chapter, the theory of the moving trihedron whose axes lie in the tangent, principal normal, and binormal of a given twisted curve, and by means of which the properties of many curves associated point for point with the given one can be very simply deduced.

Chapter II is in many respects a calculus of variations chapter. The author begins by showing how the binormal and first curvature κ of a twisted curve can be defined by the first variation of the length integral along a curve, and considers the more general problem suggested by Radon of minimizing an integral of the form $I = \int \varphi(\kappa) ds$ where s is the length of arc. He then studies the isoperimetric properties of the circle, proving that the circle is actually the curve of given length enclosing a maximum area by the methods of Frobenius and Hurwitz. One wonders a little at this point why this material is introduced here. But it turns out that in Chapter VI the author gives two very interesting generalizations, for minimal surfaces and for the sphere, of the formula which is fundamental in Frobenius' proof, and the more general results of the very elegant proof of Hurwitz are also applied in a later section. The last three sections of the chapter are devoted to two theorems of Schwarz concerning minimum properties of twisted curves with constant curvature $\kappa = 1$.

In the seventy pages of Chapters III and IV the author presents more of the theory of surfaces than I should have thought possible in such a limited space. In the former of them one finds the theories of lines of curvature, asymptotic lines and conjugate systems, the Gaussian sphere, the formulas of Gauss and Codazzi, and related subjects. The latter is devoted to the elements of the theory of applicable surfaces, geodesics and related coordinate systems, Beltrami's differential invariants, isothermal parameters and conformal representations of one surface on another. Although the presentation is concise it seems to me unusually readable and interesting.

I have mentioned already the character of the contents of Chapter V, which is one of the most interesting in the book. There are in all some nine macroscopic theorems, six concerning oval surfaces, two concerning closed surfaces, and one concerning surfaces of constant curvature $K = -1$. In § 86 the author describes in some detail a problem which is still unsolved. On every geodesic line there corresponds in general to every point A a conjugate point A'. Every arc AB not containing A' is shorter than all other arcs near it joining A and B, while arcs AB containing A' between A and B do not have this property. On a sphere the geodesics are great circles and the conjugate point A' is diametrically opposite to A on the surface. The problem of § 86 is to determine the surfaces on which the lengths of the geodesic arcs AA' are all the same. It seems likely that these surfaces are necessarily spheres, but the author at present knows no complete proof of this surmise. He describes in very interesting fashion the results attained and the difficulties remaining to be overcome.

Chapter VI is entitled "Extreme bei Flächen" and contains an introduction to the theory of minimal surfaces and a study of the isoperimetric properties of the sphere. For minimal surfaces the equations of Weierstrass and Study are derived, besides Schwartz's formula expressing the area of a minimal surface as a line integral along its contour, and the necessary condition for a minimum which he deduced from the second variation of the area integral on the surface and which corresponds to Jacobi's condition in the calculus of variations. A discussion of the problem of the determination of a minimal surface by a curve and a strip of normals on the surface, with some allied theorems, is also given. Among all surfaces with given area the sphere is the one which encloses a maximum volume, but it is not a simple matter to prove this property. The author uses a process of symmetrization of oval surfaces due to Steiner in order to prove that the maximizing surface if it exists must be a sphere. But there still remains the necessity of proving that a minimizing surface actually exists. Existence proofs have been made by Schwartz (1884), Minkowski (1903), Blaschke (1916), and Gross (1917). The proof here given is the last one, which is a modification of that of Blaschke. In the course of this proof the relation $O^3 - 36\pi V^2 \geqq 0$ is deduced for the surface area O and the volume V of an oval surface, the equality holding only when the surface is a sphere. This is a generalization of a similar relation $L^2 - 4\pi F \geqq 0$ used in an earlier chapter in the Frobenius proof of the isoperimetric property of a circle, L being the length of a plane oval and F the area which it encloses. As a second generalization the author shows also that Frobenius' formula holds for a simply closed curve in space provided that L is the length of the curve and that F is interpreted as the surface area of the minimal surface passing through the curve.

The final chapter of the book is devoted to line geometry, especially the theory of ruled surfaces and congruences. Every straight line is uniquely determined by a unit vector $\boldsymbol{a}$ and a second vector $\boldsymbol{b}$ orthogonal to the first, by means of the equations $\boldsymbol{x} \times \boldsymbol{a} = \bar{\boldsymbol{a}}$, where $\boldsymbol{x} = (\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3)$ is a variable point on the line. By the equation $\mathfrak{A} = \boldsymbol{a} + \epsilon\bar{\boldsymbol{a}}$ the straight line furthermore determines a unique vector $\mathfrak{A}$ with components in the linear algebra of Clifford whose elements have the form $\boldsymbol{a} + \epsilon\boldsymbol{b}$, $\boldsymbol{a}$ and $\boldsymbol{b}$ being real numbers and ϵ a unit such that $\epsilon^2 = 0$. It is readily provable that every such vector $\mathfrak{A}$ is a unit vector, and conversely every unit vector $\mathfrak{A}^2 = (\boldsymbol{a} + \epsilon\boldsymbol{b})^2 = 1$ has its real vectors $\boldsymbol{a}$, $\bar{\boldsymbol{a}}$ satisfying $\boldsymbol{a}^2 = 1$, $\boldsymbol{a}\bar{\boldsymbol{a}} = 0$, and therefore determines uniquely a straight line in space. By this most interesting device of Study the lines of space are set into one-to-one correspondence with the points on a unit sphere in the Clifford space, the Clifford space being the totality of points $\mathfrak{A} = (\mathfrak{A}_1, \mathfrak{A}_2, \mathfrak{A}_3)$ whose coordinates $\mathfrak{A}_1$, $\mathfrak{A}_2$, $\mathfrak{A}_3$ are numbers in the Clifford algebra. A rotation in the Clifford space determines uniquely a motion in the original $\boldsymbol{x}$-space, and conversely, so that the determination of the invariants under rotations in the former is equivalent to the determination of invariants under the group of motions in the latter. A ruled surface is now defined by a vector $\mathfrak{A}(\boldsymbol{t})$ whose elements are functions of a single parameter t, and a congruence by a vector $\mathfrak{A}(\boldsymbol{u}, \boldsymbol{v})$ depending upon two parameters. It is altogether fascinating to see how the usual properties of these surfaces and congruences fall out naturally and readily under this new aspect.

G. A. BLISS

SHORTER NOTICES

Introduction to the Mathematical Theory of the Conduction of Heat in Solids. By H. S. Carslaw. Second edition, completely revised. London, Macmillan and Co., 1921. xii + 268 pp.

The first edition of Professor Carslaw's work appeared in 1906 under the title *Introduction to the Theory of Fourier's Series and Integrals and the Mathematical Theory of the Conduction of Heat.* This book, which was reviewed in volume 15 of this BULLETIN, was divided into two parts corresponding to the two principal topics in the above title. It had been out of print for some time, and the author wisely decided to rewrite the entire book in such fashion as to take account of the progress that has been made in the theories involved since the appearance of the first edition. The very considerable expansion of size inevitably involved in such a rewriting has led naturally to the replacing of the two parts of the first edition by the two volumes of the second edition.

The first volume, which was entitled *Introduction to the Theory of Fourier's Series and Integrals*, was reviewed in volume 28 of this BULLETIN. As pointed out in that review, it showed extensive rewriting and considerable expansion as compared with the first part of the original edition. The present volume likewise exhibits considerable alteration from the second part of the first edition. Chapters I–VI, which contain substantially the same material as the corresponding chapters of the first edition, have been extensively revised; chapters VII–X contain much new material; chapters XI and XII are entirely new.

The first edition of the book was particularly noteworthy from the fact that it was the first work in English in which a large group of the standard problems that arise in the theory of the conduction of heat were dealt with systematically in a rigorous fashion. In addition to this group, however, certain other problems were introduced where the possibility of a rigorous treatment was merely indicated instead of being carried out. This was probably due partly to considerations of space, and partly to a desire to keep the book sufficiently elementary to be palatable to the student of applied mathematics, the omitted proofs being in many cases long and difficult. In some cases the omission was perhaps due to the fact that no completely rigorous discussion was available in the literature. In the present edition such of the missing proofs as are not unusually lengthy are supplied, as for example in sections 17 and 31; in most other instances, where the complete discussion is not given, suitable references are furnished which will enable the reader, if so disposed, to find a rigorous discussion in the literature. One rather noteworthy exception to this rule may be found in section 51, where the possibility of the expansion of an arbitrary function of three variables in a triple Fourier's series is assumed. The omission of any reference in this case is quite understandable, as in spite of the very extensive literature on the ordinary Fourier's series, the reviewer is unaware of any discussion of the triple Fourier's series in existing literature which would supply the missing proof in the section mentioned.

Such a discussion will be shortly available, however, in a paper * by Miss Bess Eversull, which is to appear in the ANNALS OF MATHEMATICS.

Quite a little of the new material to be found in the present edition of Professor Carslaw's book is based on his own contributions to the subject since the appearance of the first edition. Thus for example the discussion in section 90 of the flow of heat in a wedge is based on a paper by the author in the PROCEEDINGS OF THE LONDON SOCIETY ((2), vol. 8 (1909–10)). Also chapter XI, dealing with the use of contour integrals in the solution of the equation of conduction, is based on papers by the author in the PHILOSOPHICAL MAGAZINE ((6), vol. 39 (1920)) and the PROCEEDINGS OF THE CAMBRIDGE PHILOSOPHICAL SOCIETY (vol. 20 (1921)).

Chapter XII, which deals with the use of integral equations in the solution of the equation of conduction, is only a very brief sketch of this subject. It was doubtless introduced mainly for the purpose of acquainting the student of applied mathematics with the possibilities in this direction, and in this manner stimulating him to a study of the works that deal with the subject in a more extensive manner. A list of such works is given in a footnote at the beginning of the chapter.

The bibliography of works on the conduction of heat, found in Appendix II, has been brought up to date by the addition of titles of articles that have appeared since the publication of the first edition. The added titles for this period of fifteen years occupy three pages and form approximately one-fourth of the entire list, which corresponds to a period of a century. Thus we have a rough index of the mathematical activity in this particular field during recent years.

CHARLES N. MOORE

Praxis der Gleichungen. By C. Runge. Zweite, verbesserte Auflage. Berlin and Leipzig, Vereinigung wissenschaftlicher Verleger, 1921. 2 + 172 pp.

This book, printed in 1900 as number XIV of the SAMMLUNG SCHUBERT, now appears in larger format and better type as one of the new GÖSCHENS LEHRBÜCHEREI,—the second of the first group (*Reine Mathematik*). Changes have been few and unimportant. The reviewer noticed a paragraph at the foot of page 98 and a figure between pages 106 and 107, neither of which appears in the first edition, and an index has been added. Two errata have been carried over from the first edition: in line 2 from the bottom of page 38, for a_{14}, read a_{14}/a_{11}; in line 3 from the bottom of page 87, for 3217.18, read 5217.18. A new misprint is the omission of the i in the exponent of e in line 11 of page 165. But, considering the nature of the contents, there are on the whole remarkably few typographical errors.

This volume has become such a classic that it is scarcely necessary to describe it more fully than to say it gives in considerable detail, illustrated by numerous examples, the actual processes to be followed in solving numerical equations where the roots are to be found to a high degree of accuracy. Some theory is given, culminating in Sturm's Theorem, but the emphasis is laid on actual computations. It is to be hoped that sometime an English translation will make this work even more accessible to our students. A study of it is most heartily recommended to the mathematician who is interested in the applications of theory to numerical practice, to the applied mathematician who has to perform computations, and to the teacher of secondary mathematics who thinks that with Horner's method the last word on the solution of equations has been said.

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* This paper was presented to this Society at its meeting in Chicago, April 14, 1922. See this BULLETIN, vol. 28 (1922), p. 289.

ELIJAH SWIFT

Les Applications Élémentaires des Fonctions Hyperboliques à la Science de l'Ingénieur Électricien. By A. E. Kennelly. Paris, Gauthier-Villars, 1922. vii + 151 pp.

This book covers the subject of a series of lectures delivered in France by the author as exchange professor, representing a group of American universities. The same topic has been covered by Professor Kennelly in a previous work (in English) published in 1912 by the University of London Press. The differences are so slight that an American student would naturally prefer the one written in his own language.

The aim of both books is to spread more widely the fact that cable problems are best understood in terms of hyperbolic functions,—of real variables for the case of direct current with distributed leakage, of complex variables in the case of alternating currents with distributed earth or parallel-wire capacitance. This aim is very heartily to be commended. The author's method of solving alternating current problems by considering the corresponding problem with direct current, and then by replacing resistances by complex impedances is also easily the best. In some text-books this method is not used as consistently and thoroughly as it should be. It enables the principles of the wave-filter, for example, to be stated very easily by means of the two principles of the addition of impedances (admittances) of series (parallel) conductors. It is probable, however, that this method finds its way more frequently into lectures than into text-books. In this connection the reviewer would advise a more frequent use of four-terminal impedances (current between AB in terms of voltage drop from C to D).

Either of the two books would also be a useful source of problems in the elementary theory of the complex variable.

In some ways the book under review suffers from condensation. In particular, the remark that the hyperbolic angle of a sector can be measured, not only by twice its area, but also "par la longueur de l'arc hyperbolique AP, rapportée à la longueur croissante du vecteur OP" is not clear until one observes, in the earlier book, that by this is meant the integral with respect to the arc of the reciprocal of the radius.

The book under review contains a brief study of transient currents which is not contained in the earlier work.

Finally mathematicians may have their attention called to Appendix F of the earlier book in which is shown an interesting relation between continued fractions of a certain class and hyperbolic functions. The student is referred to a paper by the author in the ANNALS OF MATHEMATICS, vol. 9 (1908), p. 85 et seq.

P. J. DANIELL

Cours de Physique Mathématique de la Faculté des Sciences. Compléments au Tome III. *Conciliation du Véritable Déterminisme Mécanique avec l'Existence de la Vie et de la Liberté Morale.* Par J. Boussinesq. Paris, Gauthier-Villars, 1922. xlviii + 217 pp.

Strange title for the third volume of a book on Mathematical Physics! Strange assertion in a strange title! Yet appearing in all simplicity in the closing volume of the new edition of an older publication, now the mature judgment of a distinguished scientist over eighty years of age, the affirmation in fact of a first statement made forty-five years ago! Happy the person who can more confidently state at eighty years of age what he thought was true when he was thirty-five! Member of the Institut, author of many treatises, many memoirs, and minor papers, professor of mathematical physics and of the theory of probabilities—who has a better right to state what he finds, with confidence and serenity?

What is the startling assertion? It is very simple and yet it makes much of scientific philosophy rock preparatory to its fall in ruins. This assertion is that mechanics declares the necessity of a directive principle which is outside mechanics in order to account for the phenomena of the universe. What! Can it be that the most definite of all the branches of physics, itself the most definite and well-understood of the sciences, the branch which even gives its name to that mode of thinking which passes among some scientists as the only straight and narrow path to truth—the mechanistic philosophy—can it be that this discipline itself has turned traitor to the cause, and gone over to the camp of the enemy where freedom smiles, where there are directive principles, entelechies, even perhaps real mind? Is this the outcome of a lifetime study of the laws of mechanics and probabilities? Yet so it is. And indeed he is not without very respectable company in these opening years of a new century, now almost a quarter gone.

The investigation of M. Boussinesq has all the clarity of French mathematics, is easy to read, is not—as he says distinctly—a metaphysical argument, but is the unescapable conclusion of a mathematical consideration of the laws of motion. It is simply this in brief: the laws of motion are stated always in the form of differential equations. These have in many cases, when we come to integrate them, ambiguities in the integrals, due to the fact that the characteristic curves of integration are bifurcated at certain points and also to the fact that these curves often have envelop curves that represent the singular solutions. A particle traveling on a characteristic curve—to take the simplest case—may follow the envelop, for each characteristic curve somewhere is tangent to the enveloping curve. What determines then the path chosen in these cases, since the laws of motion show that the paths are indeterminate, and yet the particles do follow unique paths? The only answer is that at the singular points where a choice of path is possible there must be a directive principle which selects the path. This does not affect the accelerations, therefore does not consist of a force, and hence is not to be accounted for in the energy equations. This becomes the true distinctive character of phenomena due to life, conformably to the extreme physico-chemical instability of living beings, not imitated by inanimate things. The configuration of the particles of the universe is not altogether explainable by the previous configurations. There were in some situations alternative possibilities, and the laws of nature do not explain why one choice was made rather than the other,—in fact if this could be explained, there would have been no choice and no singularity in the characteristic curves.

The idea that all the phenomena of physics and chemistry can be reduced to modes of motion vanished long ago, for it became evident that the "field" played a very important part in the explanation of phenomena studied by the physicist and chemist, and the field again is described by means of partial differential equations. More complicated singular solutions have to be admitted, as well as singularities of the surfaces of the integrals. This would add force to what M. Boussinesq has to say regarding total differential equations as representing phenomena. It is evident in either case that there may be situations in which the laws of physics—all of which can be stated in mathematical form—fail to account for the following events. This simple fact is enough to invalidate mechanistic hypotheses. For these hypotheses have for their specific aim to do without any other laws than those of physics.

Boussinesq calls attention to a memoir of Poisson's which contains the incipient notion of a directive principle, arising from a consideration of singular solutions. In his *Traité de Mécanique* however he does not mention the matter. Again Duhamel reproduces Poisson's example but does not make much of it. Cournot also considered the matter, but evidently was prejudiced in favor of determinism and sought to find justification for throwing out the singular solutions and the bifurcated paths. Everything considered, there seems to be demanded a dynamics of the directive agencies, which will lead directly from the mechanics of forces to the activities of life, mind, and society. Among the principles for such a science are the principle of simplicity, and that of unity, the principle of variation, and of continuity, the principle of economy or least action. Among the problems he hopes this new science may some day solve is that of the marvelous adaptations of organ to function, of species to environment, usually dismissed with the mere statement that they are necessary for existence.

The volume has several long complementary notes, related to the main memoir. Such topics are the analogy of the mechanism of life with wave-motion, the dissipation of energy and the reversal of purely mechanical movements, the rôle and legitimacy of geometric intuition, the application of the "threshold" of sensation to a possible theory of certain quanta, and a consideration of the impossibility of knowledge of intramolecular phenomena.

This volume is most interesting for those who have philosophic instincts, not only for its positive contribution, but for the fact that it opens the way to a clearer understanding of the freedom that is found to lurk even in mathematical equations. In these days when the pendulum is swinging towards a better interpretation of the facts of life and mind, it is well to comprehend just wherein real determination finds its sphere.

James Byrnie Shaw

NOTES

The following university courses in mathematics are announced for the academic year 1923–1924:

University of Chicago.—Courses which continue for more than one quarter are indicated with Roman numerals, as I, II, III, IV. By Professor E. H. Moore: Hermitian matrices in General Analysis, I, II, III, IV; Vectors, matrices, and quaternions.—By Professor L. E. Dickson: Hypercomplex numbers, I, II; Theory of equations.—By Professor H. E. Slaught: Differential equations; Definite integrals; Elliptic integrals; Advanced calculus.—By Professor G. A. Bliss: Theory of functions of a real variable; Calculus of variations; Integral equations; Advanced calculus.—By Professor E. J. Wilczynski: Seminar on Geometry; Metric differential geometry; Solid analytic geometry.—By Professor F. R. Moulton: Modern theories of analytic differential equations, I, II; Advanced ballistics, I, II.—By Professor W. D. MacMillan: Analytic mechanics, I, II; Celestial mechanics; Dynamics of rigid bodies; Theory of the potential.—By Professor A. C. Lunn: Vector analysis; Applications of vector analysis in the theory of electromagnetism; Thermodynamics; Vector analysis in Riemann-Einstein space.—By Professor J. W. A. Young: Selected topics in mathematics.—By Dr. Mayme I. Logsdon: Theory of functions of a complex variable; Introduction to higher algebra. Courses in research are also offered by Professor Moore in Foundations of mathematics and in General Analysis, by Professor Bliss in Analysis, by Professor Dickson in Algebra and Theory of numbers, by Professor Wilczynski in Geometry, and by Professor Lunn in Applied mathematics.

Columbia University.—By Professor T. S. Fiske: Differential equations.—By Professor F. N. Cole: Theory of groups (first term).—By Professor D. E. Smith: History of mathematics; Practicum in the history of mathematics.—By Professor C. J. Keyser: Modern theories in geometry (first term); Introduction to mathematical philosophy (first term).—By Professor Edward Kasner: Einstein's theory of gravitation.—By Professor W. B. Fite: Infinite series (second term).—By Professor J. F. Ritt: Elliptic functions (first term); Analytic theory of numbers (second term).—By Dr. G. A. Pfeiffer: Topics in projective geometry (second term).—By Dr. Jesse Douglas: Topics in higher geometry (second term).

Cornell University.—By Professor J. H. Tanner: Mathematics of finance.—By Professor Virgil Snyder: Algebraic geometry.—By Professor F. R. Sharpe: Hydrodynamics and Elasticity.—By Professor Arthur Ranum: Line geometry.—By Professor W. B. Carver: Advanced calculus. —By Professor D. C. Gillespie: Theory of functions of a complex variable. —By Professor W. A. Hurwitz: Differential equations of mathematical physics.—By Professor C. F. Craig: Projective geometry.—By Professor F. W. Owens: Advanced analytic geometry.—By Professor H. M. Morse: Analysis situs (first term); The restricted problem of three bodies (second term); Elementary differential equations.—By Dr. G. M. Robison: Calculus of variations (first term); Infinite series (second term).—By Mr. D. S. Morse: Modern higher algebra.

HARVARD UNIVERSITY.—By Professor W. F. Osgood: Advanced calculus; Theory of functions (second course).—By Professor J. L. Coolidge: Line geometry (first half-year); Probability (second half-year); Subject matter of elementary mathematics (first half year); Kinematics (second half-year).—By Professor G. D. Birkhoff: Space, time, and relativity (first half-year); Advanced dynamics and quantum theory (second half-year).—By Professor E. V. Huntington: The fundamental concepts of mathematics (first half-year).—By Professor O. D. Kellogg: Dynamics (second course); Theory of potential functions (first half-year); Theory of point sets (second half-year).—By Professor W. C. Graustein: Modern geometry; Projective geometry (first half-year); Geometrical transformations (second half-year).—By Dr. J. L. Walsh: The partial differential equations of mathematical physics (second half-year).—By Dr. Philip Franklin: The analytical theory of heat and problems in elastic vibrations (second half-year); Relativity, advanced course (second half-year). There will also be a seminar in analysis conducted by Dr. Walsh and Dr. Franklin, and the following courses of research: Topics in the theory of functions, Professor Osgood; Topics in postulate-theory, Professor Huntington; Topics in geometry, Professor Coolidge; Topics in the theory of potential functions, Professor Kellogg; Topics in the theory of differential equations, Professor Birkhoff; Topics in geometry, Professor Graustein.

UNIVERSITY OF ILLINOIS.—By Professor E. J. Townsend: Functions of a complex variable; Differential equations and advanced calculus.—By Professor G. A. Miller: Theory of groups; Theory of equations and determinants (first semester).—By Professor J. B. Shaw: Vector methods. —By Professor A. B. Coble: Projective geometry.—By Professor R. D. Carmichael: Linear difference equations.—By Professor A. Emch: Algebraic surfaces; Constructive and projective geometry (second semester). —By Professor A. R. Crathorne: Statistics (first semester); Actuarial theory.—By Professor G. E. Wahlin: Calculus of variations (second semester).—By Professor A. J. Kempner: Theory of numbers.—By Professor H. Blumberg: Graphical and numerical methods (second semester); Introduction to modern mathematics.—By Professor E. B. Lytle: Teacher's course (first semester); Fundamental concepts of mathematics (second semester).

MASSACHUSETTS INSTITUTE OF TECHNOLOGY.—By Professor F. S. Woods: Advanced calculus and differential equations; Higher geometry. —By Professor C. L. E. Moore: Theoretical aeronautics; Rigid dynamics. —By Professor H. B. Phillips: Thermodynamics; Statistical mechanics; Quantum theory; Theory of the gyroscope.—By Professor Joseph Lipka: Analytical mechanics; Mathematical laboratory.—By Professor F. L. Hitchcock: Application of mathematics to Chemistry.—By Dr. George Rutledge: Modern algebra; Theory of functions.—By Dr. J. S. Taylor: Mathematics of investment.—By Dr. N. Wiener: Fourier's series and integral equations.—By Dr. S. D. Zeldin: Vector analysis.

UNIVERSITY OF PENNSYLVANIA:—By Professor E. S. Crawley: Higher plane curves.—By Professor G. H. Hallett: Infinite series and products (first term); The theory of functions of a complex variable (second term).

—By Professor O. E. Glenn: The theory of invariants.—By Professor F. H. Safford: The mathematical theory of elasticity.—By Professor G. G. Chambers: Synthetic projective geometry.—By Professor H. H. Mitchell: The analytic theory of numbers.—By Professor M. J. Babb: Introduction to the theory of numbers.—By Professor F. W. Beal: Linear differential equations of the second order (first term); Advanced calculus (second term).—By Professor J. R. Kline: Pointset theory (first term); Integral equations (second term).

PRINCETON UNIVERSITY:—By Professor J. H. M. Wedderburn: Complex variables.—By Professor L. P. Eisenhart: Differential geometry.—By Dr. C. E. Hille: Advanced course in analysis.—By Dr. C. C. MacDuffee: Algebraic invariants.—By Dr. B. Kerekjarto: Analysis situs.—By Professor O. Veblen: Seminar on mathematical physics.

At the request of the Committee on Endowment of this Society, Professor J. W. Young has consented to take charge of the general publicity for that committee.

During the first semester of the academic year 1923–24, Professor B. A. Bernstein of the University of California will be on leave of absence, and will reside in New Haven, Conn. During his absence from California, Professor D. N. Lehmer will act in his stead as secretary of the San Francisco Section of the Society.

The Polish Academy of Sciences and Letters of Cracow celebrated its fiftieth anniversary on June 16, 1923.

By a special arrangement recently made, members of the Deutsche Mathematiker-Vereinigung who contribute two dollars ($2.00) yearly will receive the JAHRESBERICHT, complete in both parts. Membership is open to mathematical departments as units, as well as to individuals. The usual membership with the regular dues, which are at present 600 marks per year, carries with it merely the subscription to the second part of the JAHRESBERICHT, containing business notes, reports of meetings, reviews, problems and solutions, etc.; it is to be hoped that those joining on this basis will voluntarily add a small subscription to cover postage. Applications for membership should be sent to Professor L. Bieberbach, Berlin-Schmargendorf, Marienbaderstrasse 9.

Mr. L. J. Mordell, of Cambridge University, has been appointed a Fielden professor of pure mathematics at the University of Manchester. Professor Mordell (cf. this BULLETIN, April, 1923, p. 189) will give two courses this summer at the University of Chicago.

Dr. W. A. Hamilton will be on the staff of the University of Wisconsin for the coming year as Lecturer in Mathematics. For the past ten years he has been head of the department of Mathematics in Beloit College, and for the past five years Registrar.

Assistant Professor E. P. Lane, of the University of Wisconsin, has been appointed assistant professor of mathematics at the University of Chicago.

Assistant Professor W. L. Crum, of Yale University, has been appointed assistant professor of statistics in the department of economics at Harvard University.

Professor Charles Niven, of the University of Aberdeen, died May 11, 1923, at the age of seventy-eight years.

NEW PUBLICATIONS

I. PURE MATHEMATICS

Archimedes. Kugel und Zylinder. (Ostwald's Klassiker der Exakten Wissenschaften, Nr. 202.) Leipzig, Akademische Verlagsgesellschaft, 1922. 80 pp.

Bally (E.). Principes et premiers développements de géométrie synthétique générale moderne. Paris, Gauthier-Villars, 1922. 8 + 218 pp.

Browne (R. T.). The mystery of space: a study of the hyperspace movement in the light of the evolution of new psychic faculties, and an inquiry into the genesis and essential nature of space. London, Routledge, 1921. 8vo.

de Comberousse (C.). See Rouché (E.).

Cuny (G.). Un théorème de géométrie et ses applications. Paris, Vuibert, 1923. 6 + 102 pp.

Czuber (E.). Die philosophischen Grundlagen der Wahrscheinlichkeitsrechnung. (Wissenschaft und Hypothese, XXIV.) Leipzig, Teubner, 1923. 8 + 343 pp.

Encyklopädie der mathematischen Wissenschaften. Band III 1, Heft 9: E. Steinitz, Polyeder und Raumeinteilung. Leipzig, Teubner, 1922.

Ferguson (D. F.) and Piggott (H. E.). Areas and volumes; their accurate and approximate determination. London, Constable, 1923. 8vo. 7 + 88 pp.

de Fermat (P.). Einführung in die ebenen und körperlichen Oerter. (Ostwald's Klassiker der Exakten Wissenschaften, Nr. 208.) Leipzig, Akademische Verlagsgesellschaft, 1923. 22 pp.

Gifford (E.). Natural tangents. To every second of arc and eight places of decimals. London and New York, 1922. 270 pp.

Happach (V.). Ausgleichungsrechnung nach der Methode der kleinsten Quadrate. Leipzig, Teubner, 1923. 74 pp.

Hecke (E.). Vorlesungen über die Theorie der algebraischen Zahlen. Leipzig, Akademische Verlagsgesellschaft, 1923. 8 + 266 pp.

Levi Civita (T.). See Ricci (G.).

Lietzmann (W.). Erkenntnislehre im mathematischen Unterricht der Oberklassen. Charlottenburg, Mundus Verlagsanstalt, 1921. 68 pp.

Mills (C. N.). Introduction to plane analytical geometry and differential calculus. Philadelphia, Blakiston, 1922. 11 + 131 pp. $1.50

Piggott (H. E.). See Ferguson (D. F.).

Pincherle (S.). Gli elementi della teoria delle funzioni analitiche. Parte I. Bologna, Zanichelli, 1922. 8vo. 8 + 401 pp.

Ricci (G.) et Levi Civita (T.). Méthodes de calcul différentiel absolu et leurs applications. (Réimpression, Mathematische Annalen, tome 54.) (Collection de Monographies scientifiques étrangères.) Paris, A. Blanchard, 1923. 77 pp.

Rose (W. N.). Mathematics for engineers. 2d edition. Part I. London, Chapman and Hall, 1922. 14 + 514 pp.

Rouché (E.) et de Comberousse (C.). Traité de géométrie. 1re partie: Géométrie plane. 2e partie: Géométrie dans l'espace. Paris, Gauthier-Villars, 1922. 544 + 18 + 664 pp.

Rutgers (J. G.). Inleiding tot de analytische meetkunde. Eerste deel: Het platte vlak. Groningen, Noordhoff, 1923.

Sanguet (J. L.). Tables trigonométriques centésimales. 2e édition. Paris, Gauthier-Villars, 1922.

Smith (H. B.). A first book in logic. New York, Harpers, 1922. 172 pp.

Speiser (A.). Theorie der Gruppen von endlicher Ordnung. Berlin, Springer, 1923. 8 + 194 pp.

Steinitz (E.). See Encyklopädie.

Study (E.). Mathematik und Physik. Eine Erkenntnistheoretische Untersuchung. Braunschweig, Vieweg, 1923. 31 pp.

de Tannenberg (W.). Calcul des erreurs absolus et des erreurs relatifs. Paris, Vuibert, 1922. 35 pp.

de Vries (H.). Beknopt leerboek der projectieve meetkunde. Groningen, Noordhoff, 1923.

Whipple (G. C.). Vital statistics. An introduction to the science of demography. 2d edition, revised and brought up to date. New York, Wiley, 1923. 576 pp. $5.00

Willers (A.). Numerische Integration. (Sammlung Göschen.) Berlin, Vereinigung wissenschaftlicher Verleger, 1923. 116 pp.

Wittgenstein (L.). Tractatus logico-philosophicus. London, Kegan, Paul and Company, and New York, Harcourt, Brace and Company, 1922. 189 pp.

Wyckoff (R. W. G.). The analytical expression of the results of the theory of space groups. Washington, Carnegie Institution, 1922. 8vo. 7 + 180 pp.

II. APPLIED MATHEMATICS

Blattner (E.). Lehrbuch der Elektrotechnik. 4te Auflage. Teil 1. Berlin, K. J. Wyss Erben, 1922. 9 + 423 pp.

Booth (H.). Aeroplane performance calculations. London, Chapman and Hall, 1921. 8vo. 208 pp.

Boscovich (R. J.). A theory of natural philosophy. Latin-English edition. From the text of the first Venetian edition published under the personal superintendence of the author in 1763. Chicago and London, Open Court, 1922. 19 + 470 pp.

Brahe (T.). Tychonis Brahi opera omnia. Edidit I. L. E. Dreyer. Tomus IV. Hauniæ, Libraria Gyldendaliana, 1922. 148 pp.

Brill (A.). Das Relativitätsprinzip. Eine Einführung in die Theorie. 4te Auflage. Leipzig, 1920.

de Broglie (M.). Exposé concernant les résultats actuels relatifs aux éléments isotopes. Paris, Hermann, 1922. 15 pp.

Buhl (A.). Les théories eisteiniennes et les principes du calcul intégral. Paris, Gauthier-Villars, 1922. 16 pp.

Butler (E.). Modern pumping and hydraulic machinery. 2d edition, revised. London, Griffin, 1922. 493 pp.

Chatley (H.). A text-book of aeronautical engineering. The problem of flight. 3d edition. London, Griffin, 1921. 12 + 150 pp.

Dreyer (I. L. E.). See Brahe (T.).

Gray (W. M.). Naval architecture. A first course in ship calculations. London, Methuen, 1923. 5 + 201 pp.

Grübler (M.). Lehrbuch der technischen Mechanik. Band 1: Bewegungslehre. 2te, verbesserte Auflage. Band 3: Dynamik starrer Körper. Berlin, Springer, 1921.

Haag (J.). Cours complet de mathématiques spéciales. Exercices du Tome III (Mécanique). Paris, Gauthier-Villars, 1922. 202 pp.

Lichtenstein (L.). Astronomie und Mathematik in ihrer Wechselwirkung. Mathematische Probleme in der Theorie der Figur der Himmelskörper. Leipzig, Hirzel, 1923. 8vo. 8 + 97 pp.

Pfaff (A.). Für und gegen das Einsteinschen Prinzip. Diessen vor München, Huber, 1921. 44 pp.

Richter (G.). Kritik der Relativitätstheorie Einsteins. Leipzig, Hillman, 1921.

Schwassmann (A.). Relativitätstheorie und Astronomie, auf wissenschaftlicher Grundlage gemeinverständlich dargestellt. Hamburg, Grand, 1921. 8vo. 34 pp.

See (T. J. J.). Electrodynamic wave-theory. Volume 2: New theory of the æther. London, Wheldon and Wesley, 1922.

de Sparre (—.). Théorie analytique des turbines centripètes et centrifuges à reaction. Louvain, Ceuterick, et Paris, Gauthier-Villars, 1922. 42 pp.

Stark (J.). Die gegenwärtige Krisis in der deutschen Physik. Leipzig, Barth, 1922. 6 + 32 pp.

Swick (C. H.). Modern methods for measuring the intensity of gravity. Washington, Government Printing Office, 1921. 8vo. 96 pp.

Tolle (M.). Regelung der Kraftmaschinen. 3te, verbesserte und vermehrte Auflage. Berlin, Springer, 1921. 12 + 890 pp.

Vincent (M.). La gravitation thermodynamique et ses conséquences. La précession des équinoxes et l'inversion océanique. Premiers éléments de thermodynamique céleste. Paris, Librairie Fischbacher, 1921. 16mo. 204 pp.

Walsh (J. W. T.). The elementary principles of lighting and photometry. London, Methuen, 1923. 16 + 220 pp.

Weber (C.). Die Lehre der Drehungsfestigkeit. Berlin, Verlag des Vereines deutscher Ingenieure, 1921. 70 pp.

Whyte (C.). Our solar system and the stellar universe. Ten popular lectures. London, Griffin, 1923. 8vo. 11 + 234 pp. + 18 plates.

Wittenbauer (F.). Aufgaben aus der technischen Mechanik. 3te, verbesserte und vermehrte Auflage. Band 3: Flüssigkeiten und Gase. Berlin, Springer, 1923. 8 + 390 pp.

Zimm (W.). Ueber die Strömungsvorgänge im freien Luftstrahl. Berlin, Verlag des Vereines deutscher Ingenieure, 1921. 36 pp.

ON THE RIEMANN ZETA FUNCTION *

BY C. F. CRAIG

In this note a functional relation for the Riemann zeta function is established, and its relation to the well known formula given by Riemann is pointed out. It is believed that the relation is new.

Recall that the even elliptic theta constants are

$$(1)\qquad \begin{cases} \vartheta_{00}(x) = 1 + 2\sum\limits_{n=1}^{\infty} q^{4n^2} = 1 + 2\phi_{00}(x), \\ \vartheta_{01}(x) = 2\sum\limits_{n=1}^{\infty} q^{(2n-1)^2} = 2\phi_{01}(x), \\ \vartheta_{10}(x) = 1 - 2\sum\limits_{n=1}^{\infty} (-1)^{n-1} q^{4n^2} = 1 - 2\phi_{10}(x), \end{cases}$$

with $q = e^{-(\pi/4)x^2}$. These series converge uniformly and absolutely for $0 < \delta \leqq x$. Furthermore

$$\vartheta_{00}\left(\frac{1}{x}\right) = x\vartheta_{00}(x), \qquad \vartheta_{01}\left(\frac{1}{x}\right) = x\vartheta_{10}(x), \qquad \vartheta_{10}\left(\frac{1}{x}\right) = x\vartheta_{01}(x).$$

Let

$$(2)\qquad \vartheta(x) = x^{1/2}[\vartheta_{01}(x) + \vartheta_{10}(x) - \vartheta_{00}(x)];$$

then

$$(3)\qquad \vartheta\left(\frac{1}{x}\right) = \vartheta(x).$$

Also

$$1 + 2\phi_{00}\left(\frac{1}{x}\right) = x[1 + 2\phi_{00}(x)]$$

with two similar relations for ϕ_{01} and ϕ_{10}. From these follow

$$(4)\qquad \begin{cases} \lim\limits_{x\to 0} x\phi_{00}(x) = \dfrac{1}{2}, & \lim\limits_{x\to 0} x\phi_{10}(x) = 0, \\ \lim\limits_{x\to 0} x\phi_{01}(x) = \dfrac{1}{2}, & \lim\limits_{x\to 0} x^{h}\,\vartheta(x) = 0, \end{cases}$$

where h is any constant. To prove the last of the expressions

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* Presented to the Society, December 27, 1922.

(4) observe that

$$\vartheta(y) = 2y^{1/2}\sum_1^\infty Q_n(y),$$

where $Q_n(y) = q^{(2n-1)^2} - 2q^{4(2n-1)^2}$. For $y \geqq 1$, $Q_n(y) > 0$, and hence by (3) $\vartheta(y)$ is never negative. Also

$$Q_n(y) < e^{-(\pi/4)y^2}e^{-n\pi}, \qquad y \geqq 1,$$

and hence

$$\vartheta(y) < 2y^{1/2}e^{-(\pi/4)y^2}(1 - e^{-\pi/4})^{-1}, \qquad y \geqq 1.$$

In $x^h\vartheta(x)$ placing $x = 1/y \leqq 1$, and using (3) together with the last inequality, we find

$$|x^h\vartheta(x)| < 2(1 - e^{-\pi/4})^{-1}|y^{\frac{1}{2}-h}|e^{-(\pi/4)y^2}.$$

The last of the expressions (4) follows for $y \to \infty$.

Recall further that

$$(5) \qquad \begin{cases} \zeta(s) = 1 + \dfrac{1}{2^s} + \dfrac{1}{3^s} + \dfrac{1}{4^s} + \cdots, \\ (1 - 2^{1-s})\zeta(s) = 1 - \dfrac{1}{2^s} + \dfrac{1}{3^s} - \dfrac{1}{4^s} + \cdots, \\ (1 - 2^{-s})\zeta(s) = 1 + \dfrac{1}{3^s} + \cdots, \end{cases}$$

and that each of these series converges absolutely and uniformly for $R(s) \geqq 1 + \delta$, $R(s)$ being the real part of s. In

$$\Gamma\left(\frac{s}{2}\right) = \int_0^\infty e^{-z}z^{(s/2)-1}dz, \qquad R(s) > 0,$$

replace z by $k^2\pi x^2$ and there results

$$(6) \qquad \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)k^{-s} = 2\int_0^\infty e^{-k^2\pi x^2}x^{s-1}dx, \qquad R(s) > 0.$$

In (6) put $k = n$, multiply each side of the equality by $(-1)^{n-1}$, and sum with respect to n. This gives, on account of (5),

$$(1 - 2^{1-s})\chi(s) = \sum_{n=1}^\infty 2\int_0^\infty (-1)^{n-1}e^{-n^2\pi x^2}x^{s-1}dx, \qquad R(s) > 1,$$

where

$$\chi(s) = \pi^{-s/2}\Gamma\left(\frac{s}{2}\right)\zeta(s).$$

An easy investigation * shows that the operations of integration and summation are reversible, giving

$$(1 - 2^{1-s})\chi(s) = 2\int_0^\infty \phi_{10}(x)x^{s-1}dx, \qquad R(s) \geqq \delta + 2.$$

The use of (6) with $k = n$ and $2k = 2n - 1$ gives, on summing with respect to n,

$$\chi(s) = 2\int_0^\infty \phi_{00}(x)x^{s-1}dx, \qquad R(s) \geqq \delta + 2,$$

$$2^s(1 - 2^{-s})\chi(s) = 2\int_0^\infty \phi_{01}(x)x^{s-1}dx, \qquad R(s) \geqq \delta + 2.$$

The first of this pair of relations is Riemann's well known formula. These three equalities combine to yield, by using the relations (1) and (2),

$$(7) \qquad -(2^s - 1)(2^{1-s} - 1)\chi(s) = \int_0^\infty \vartheta(x)x^{s-\frac{1}{2}}x^{-1}dx.$$

The derivation of formula (7) is the principal object of this note. Thanks to the last of the relations (4), the integral in (7) converges for every value of s and hence the representation is valid everywhere. In (7) replace x with $1/x$, use (3) and obtain

$$-(2^s - 1)(2^{1-s} - 1)\chi(s) = \int_0^\infty \vartheta(x)x^{-(s-\frac{1}{2})}x^{-1}dx.$$

The addition of this equation to (7) gives

$$(8) \quad -2(2^s-1)(2^{1-s}-1)\chi(s) = \int_0^\infty \vartheta(x)[x^{s-\frac{1}{2}}+x^{-(s-\frac{1}{2})}]x^{-1}dx.$$

In this replace s by $1 - s$ and we deduce the well known property of $\chi(s)$, viz., $\chi(s) = \chi(1 - s)$. Since $\vartheta(x)$ is never

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* That given by Whittaker and Watson, *Modern Analysis*, 3d ed., p. 273, is readily adapted.

negative, the right hand member of (7) does not vanish for s real. Thus $\chi(s)$ has a simple pole at $s = 0$ and at $s = 1$ and has no real zeros. Recalling the location of the poles, all simple, of $\Gamma(s/2)$, we observe that $\zeta(s)$ has -2, -4, -6, $\cdots$ as its only real zeros and that these are simple.

In (8), let $s = \frac{1}{2} + it$, and we have

$$X(t) = \int_0^\infty \vartheta(x) \cos (t \log x) x^{-1} dx, \tag{9}$$

with

$$X(t) = (2^{\frac{1}{2}+it} - 1)(1 - 2^{\frac{1}{2}-it})\chi(\tfrac{1}{2} + it).$$

In (9), separate the integral into the two parts from 0 to 1 and from 1 to ∞, in the first replace x with $1/x$, employ (3), and we have

$$X(t) = 2\int_1^\infty \vartheta(x) \cos (t \log x) x^{-1} dx.$$

Now replace x by e^x, and we obtain

$$X(t) = 2\int_0^\infty \vartheta(e^x) \cos (tx) dx.$$

Observing that $X(t)$ is an entire function, we obtain the series representation

$$X(t) = \sum_{n=0}^{\infty} b_n t^{2n}. \tag{10}$$

This series has the known zeros $t = (2k\pi/\log 2) \pm i/2$, where $k = 0, \pm 1, \pm 2, \cdots$, together with an unlimited number of real zeros. So far neither the series (10) nor any of the chain of integrals leading up to it has yielded any new information relative to the verification of the hypothesis of Riemann about the nature of the zeros of $\zeta(s)$. If we get rid of the zeros of $X(t)$ corresponding to those of $(1 - 2^s)(1 - 2^{1-s})$ by employing theta relations such as $2\vartheta_{01}(2x) = \vartheta_{00}(x) - \vartheta_{10}(x)$, then (7), or any of its consequences, reduces immediately to the corresponding result in the Riemann development.

CORNELL UNIVERSITY

NOTE ON A CERTAIN TYPE OF RULED SURFACE*

BY W. C. GRAUSTEIN

In the January, 1923, number of this BULLETIN, J. K. Whittemore discusses ruled surfaces having the property that any two secondary asymptotic lines cut equal segments from the rulings. It so happens that, in investigating the determination of a surface by the linear element of its spherical representation and its total and mean curvatures, the present writer was led to consider the same class of ruled surfaces, with the results which are set forth in this note. The method of attack differs from that of Whittemore and the facts obtained overlap only in the case of the characteristic property, namely, that the rulings are parallel to a plane and the parameter of distribution is constant.

Any two secondary asymptotic lines of a ruled surface cut equal segments from the rulings if and only if the surface, when referred to its asymptotic lines as the parametric curves, admits a representation of the usual form,

$$(1) \qquad x_i = \xi_i(v) + u\eta_i(v), \qquad (i = 1, 2, 3),$$

where η_1, η_2, η_3 are the direction cosines of the ruling, η, and u is the algebraic distance along the ruling from the directrix, $\xi = \xi(v)$. Analytically, this condition amounts to demanding that $D'' = 0$. But HD'' is a quadratic polynomial in u, whose coefficients are functions of v alone. Thus the condition $D'' = 0$ gives rise to three equations, namely:

$$(2) \qquad (\eta\xi'\xi'') = 0, \qquad (\eta\eta'\xi'') + (\eta\xi'\eta'') = 0, \qquad (\eta\eta'\eta'') = 0.$$

The vanishing of the last determinant is the condition that the director cone degenerate into a plane. The spherical indicatrix, $\eta_i = \eta_i(v)$, $(i = 1, 2, 3)$, is then a circle of unit radius, in particular, the circle in which the director plane cuts the unit sphere. If we choose as the parameter v the arc of this circle, it follows that $\eta_i'' = -\eta_i$, $(i = 1, 2, 3)$, and conditions (2) become

$$(3) \qquad (\eta\xi'\xi'') = 0, \qquad (\eta\eta'\xi'') = 0.$$

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* Presented to the Society, April 28, 1923.

The first of these conditions requires merely that the directrix be a secondary asymptotic line and can always be assumed fulfilled. The second is the condition that the parameter of distribution be constant, for the value of this parameter is readily shown to be $\pm(\eta\eta'\xi')$. The characteristic property is thus established.

If we exclude the trivial case of a developable, when the surface is necessarily a plane, the directions η, ξ', η' are distinct and not parallel to a plane. Consequently, equations (3) are satisfied only if η and ξ'' are linearly dependent, that is, only if *

$$(4) \qquad \eta_i = \frac{\xi_i''}{\sqrt{\xi''|\xi''}}, \qquad (i = 1, 2, 3),$$

where there is no loss of generality in admitting merely the positive square root. Moreover, the assumption $(\xi''|\xi'') \neq 0$ is readily justified, since not all the secondary asymptotic lines can be straight and one which is not can be chosen as the directrix. If we introduce the curve y defined by the equations $y_i = \xi_i'$, $(i = 1, 2, 3)$, the representation (1) of the surface becomes

$$(5) \qquad x_i = \int y_i dv + u\eta_i, \qquad (i = 1, 2, 3).$$

Furthermore, we have, in place of (4),

$$\eta_i = \frac{y_i'}{\sqrt{y'|y'}}, \qquad (i = 1, 2, 3).$$

In other words, the circle $\eta = \eta(v)$, which is the spherical indicatrix of the surface, is also the tangent indicatrix of the curve y. This curve, then, is a plane curve. Its plane is parallel to the director plane, but not coincident with it, since otherwise (5) would represent a developable (a plane).

If $y = y(v)$ is an arbitrary plane curve not in a plane through the origin and v is the arc of its tangent indicatrix, $\eta = \eta(v)$, the general ruled surface having the property that the secondary asymptotic lines cut equal segments from the rulings is represented by equations (5).

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* If a denotes the number triple a_1, a_2, a_3, then $(a|a) = a_1^2 + a_2^2 + a_3^2$.

We next investigate under what conditions the surface is a conoid. The asymptotic line $u = u_0$ is straight only when the vector product of x_v and x_{vv} vanishes for $u = u_0$. But this vector product is equal to the scalar, $R - u$, multiplied by a non-vanishing triple, where R is the radius of curvature of the curve y. Hence it is zero for some constant value of u only if R is constant.

The surface is a conoid if and only if the curve y is a circle.

For $u = R =$ const., $x_{vv} = 0$. Hence the arc of the directrix line of the conoid can be taken proportional to v, or, since the arc v of the unit circle, $\eta = \eta(v)$, is also the corresponding angle at the center of the circle, the distance between two arbitrary points of the directrix line is proportional to the angle between the rulings through these points. In fact, if we introduce $u - R = \bar{u}$ as a new parameter, equations (5) become

$$x_i = a_i v + \bar{u}\eta_i(v), \qquad (i = 1, 2, 3), \tag{6}$$

where a_1, a_2, a_3 are constants defining a direction not parallel to the director plane.

The secondary asymptotic lines of a conoid cut equal segments from the rulings if and only if the angle through which a variable ruling turns is proportional to the distance along the directrix line through which the ruling slides.

A right conoid with this property is necessarily a right helicoid, the only minimal ruled surface. It is in this case alone that the curve y of (5) is a circle subtending at the origin a cone of revolution.

If the (x_1, x_2)-plane be taken as the director plane, η_1, η_2, η_3 can be chosen respectively as cos v, sin v, 0, and equations (5) become

$$x_1 = \int dv \int R \cos v \, dv,$$

$$x_2 = \int dv \int R \sin v \, dv,$$

$$x_3 = cv,$$

where R is an arbitrary function of v and c an arbitrary constant, not zero. Whittemore's equations (1″), taken in conjunction with his relation (6), are reducible to this form.

HARVARD UNIVERSITY

THE DIFFERENTIATION OF A FUNCTION OF A FUNCTION

BY H. S. CARSLAW

The reviewer of Rothe's interesting work *Vorlesungen über Höhere Mathematik* (1921) in this BULLETIN (vol. 28, p. 468) calls attention to the author's tentative claim that the first valid proof of the formula for the derivative of a function of a function is to be found therein, and he mentions the careful treatment of the question in Pierpont's *Theory of Functions of a Real Variable* (1905).

It is perhaps worth while to notice that Dini in his *Lezioni di Analisi Infinitesimale* (autographed edition, 1877) and Genocchi-Peano in their *Calcolo Differenziale* (1884) both gave satisfactory proofs. The treatment of Genocchi-Peano is cited and reproduced by Stolz in *Grundzüge der Differential- und Integralrechnung*, Bd. I (1893).

A proof on the same lines as that of Pierpont was given by Tannery in his *Introduction à la Théorie des Fonctions d'une Variable* (1886). See also Cesàro's *Lehrbuch der Algebraischen Analysis und der Infinitesimalrechnung*, Deutsch von Kowalewski (1904), and Kowalewski's *Grundzüge der Differential- und Integralrechnung* (1909).

It is remarkable that even the most careful English writers on the calculus have missed the defect in the proof to be found in our standard works on that subject.

THE UNIVERSITY OF SYDNEY, AUSTRALIA

ON THE RELATIVE CURVATURE OF TWO CURVES IN V_n*

BY JOSEPH LIPKA

1. *Definitions of Geodesic Curvature.* In any space of n dimensions V_n whose first fundamental form is given by †

$$(1) \qquad ds^2 = \sum_{ik} a_{ik} dx_i dx_k,$$

the geodesic curvature κ of a curve c at a point P may be defined in one of the two following ways:

(i) Draw the geodesic g tangent to c at P and on g and c lay off from P equal infinitesimal arc lengths δs; let Q and $\overline{Q}$ be the extremities of these arcs on c and g respectively; then ‡

$$(2) \qquad \kappa = \lim_{Q \to P} \frac{2Q\overline{Q}}{(\delta s)^2}.$$

(ii) Consider an infinitesimal element $PQ = \delta s$ of c and the geodesic g having this element in common with c, i.e., as Q approaches P as a limit, c and g will approach tangency at P; the immediately following elements of c and g will not in general coincide but will form at Q an infinitesimal angle $\delta\omega$; then §

$$(3) \qquad \kappa = \lim_{Q \to P} \frac{\delta\omega}{\delta s}.$$

Both these definitions lead to the same analytical expression for the geodesic curvature κ, viz.,

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* Presented to the Society, February 24, 1923.

† Throughout this paper all summations extend from 1 to n for the indicated subscripts.

‡ This definition is that given by Voss, MATHEMATISCHE ANNALEN, vol. 16. Cf. Bianchi, *Geometria differenziale*, 2d ed., vol. 1, p. 363.

§ This definition, or an equivalent one in terms of the parallelism of Levi-Civita, was given by the author in a paper, *Sulla curvatura geodetica delle linee appartenenti ad una varietà qualunque*, RENDICONTI ACCADEMIA DEI LINCEI, vol. 31 (1922).

$$\kappa = \sum_{rt} a_{rt} \left[\frac{d^2x_r}{ds^2} + \sum_{ik} \begin{Bmatrix} i\ k \\ r \end{Bmatrix} \frac{dx_i}{ds}\frac{dx_k}{ds} \right] \times \left[\frac{d^2x_t}{ds^2} + \sum_{ik} \begin{Bmatrix} i\ k \\ t \end{Bmatrix} \frac{dx_i}{ds}\frac{dx_k}{ds} \right], \tag{4}$$

where $\{{}^{i\ k}_{\ r}\}$ is the well known Christoffel symbol of the second kind.

It is the purpose of this note to generalize the above procedure by replacing the geodesic g in (i) by any other curve $\bar{c}$ tangent to c at P, and (ii) by any other curve $\bar{c}$ having an infinitesimal element PQ in common with c.

2. *Generalization of the First Definition.* We have any two curves c and $\bar{c}$ tangent at P and two equal infinitesimal arc lengths PQ and $P\bar{Q}$ ($= \delta s$) on c and $\bar{c}$ respectively. Let the coordinates of P be x_r ($r = 1, 2, \cdots, n$), those of Q and $\bar{Q}$ (developing in powers of δs) be, respectively,

$$x_r + x_r'\delta s + \tfrac{1}{2}x_r''(\delta s)^2, \qquad (r = 1, 2, \cdots, n),$$
$$x_r + x_r'\delta s + \tfrac{1}{2}\bar{x}_r''(\delta s)^2, \qquad (r = 1, 2, \cdots, n),$$

(disregarding infinitesimals of higher order than the second), where the direction of the common tangent at P is given by $x_r' = dx_r/ds$ ($r = 1, 2, \cdots, n$), and where x_r'' and $\bar{x}_r''$ are the values of d^2x_r/ds^2 computed at P for c and $\bar{c}$ respectively. The differences of these coordinates are

$$\tfrac{1}{2}(x_r'' - \bar{x}_r'')(\delta s)^2,$$

and hence we have for the distance $Q\bar{Q}$ the expression

$$(Q\bar{Q})^2 = \sum (a_{rt})_Q (x_r'' - \bar{x}_r'')(x_t'' - \bar{x}_t'')(\delta s)^4/4,$$

where $(a_{rt})_Q$ represents the values of the coefficients a_{rt} at the point Q, i.e.,

$$(a_{rt})_Q = a_{rt} + \frac{da_{rt}}{ds}\delta s + \cdots. \tag{5}$$

Therefore we have

$$\left[\lim_{Q \to P} \frac{2Q\bar{Q}}{(\delta s)^2} \right]^2 = \sum_{rt} a_{rt}(x_r'' - \bar{x}_r'')(x_t'' - \bar{x}_t''). \tag{6}$$

3. *Generalization of the Second Definition.* We have any two curves c and $\bar{c}$ having an infinitesimal element $PQ = \delta s$ in common (i.e., as $Q \rightarrow P$, the curves c and $\bar{c}$ will approach tangency at P). The immediately following elements QR and $Q\bar{R}$ of c and $\bar{c}$ will form an infinitesimal angle $\delta\omega$ at Q. Let the direction PQ have for parameters $x_r' = dx_r/ds$ ($r = 1, 2, \cdots, n$), i.e.,

$$\sum_{rt} a_{rt} x_r' x_t' = 1. \tag{7}$$

The directions of QR and $Q\bar{R}$ may be expressed by the parameters $x_r' + x_r''\delta s$, $x_r' + \bar{x}_r''\delta s$ (disregarding infinitesimals of higher order than the first), bound by the relations

$$\left\{\begin{aligned} &\sum_{rt}(a_{rt})_Q(x_r' + x_r''\delta s)(x_t' + x_t''\delta s) = 1, \\ &\sum_{rt}(a_{rt})_Q(x_r' + \bar{x}_r''\delta s)(x_t' + \bar{x}_t''\delta s) = 1. \end{aligned}\right. \tag{8}$$

The angle $\delta\omega$ between these two directions at Q is given by

$$\cos \delta\omega = \sum_{rt}(a_{rt})_Q(x_r' + x_r''\delta s)(x_t' + \bar{x}_t''\delta s).$$

Using the first identity (8), this becomes

$$\cos \delta\omega = 1 + \sum_{rt}(a_{rt})_Q(x_r' + x_r''\delta s)(\bar{x}_t'' - x_t'')\delta s. \tag{9}$$

Subtracting the identities (8), we have

$$2\sum_{rt}(a_{rt})_Q(x_r' + x_r''\delta s)(\bar{x}_t'' - x_t'')\delta s + \sum_{rt}(a_{rt})_Q(\bar{x}_r'' - x_r'')(\bar{x}_t'' - x_t'')(\delta s)^2 = 0,$$

and introducing this into (9), we find

$$\cos \delta\omega = 1 - \tfrac{1}{2}\sum_{rt}(a_{rt})_Q(x_r'' - \bar{x}_r'')(x_t'' - \bar{x}_t'')(\delta s)^2. \tag{10}$$

On the other hand,

$$\cos \delta\omega = 1 - \tfrac{1}{2}(\delta\omega)^2 + \cdots. \tag{11}$$

Comparing (10) and (11), we deduce that

$$\lim_{Q \rightarrow P}\left(\frac{\delta\omega}{\delta s}\right)^2 = \sum_{rt} a_{rt}(x_r'' - \bar{x}_r'')(x_t'' - \bar{x}_t''). \tag{12}$$

We note that the right members of (6) and (12) are identical.

4. *Relative Curvature.* The expression

$$\sum_{rt} a_{rt}(x_r'' - \bar{x}_r'')(x_t'' - \bar{x}_t'') \tag{13}$$

has an interesting geometric interpretation. Let us set

$$A_r = \sum_{ik} \left\{ \begin{matrix} i\,k \\ r \end{matrix} \right\} x_r' x_k'. \tag{14}$$

We may then write (13) in the form

$$\begin{aligned} &\sum_{rt} a_{rt}[(x_r''+A_r)-(\bar{x}_r''+A_r)][(x_t''+A_t)-(\bar{x}_t''+A_t)] \\ &= \sum_{rt} a_{rt}(x_r''+A_r)(x_t''+A_t)+\sum_{rt} a_{rt}(\bar{x}_r''+A_r)(\bar{x}_t''+A_t) \\ &\qquad -2\sum_{rt} a_{rt}(x_r''+A_r)(\bar{x}_t''+A_t). \end{aligned} \tag{15}$$

We here introduce the geodesic curvature κ as given by (4), and the direction of the principal geodesic normal to a curve c at a point P as given by the parameters *

$$\mu^{(r)} = \frac{1}{\kappa}(x_r'' + A_r), \quad (r = 1, 2, \cdots, n), \tag{16}$$

so that

$$\sum_{rt} a_{rt}(x_r'' + A_r)(\bar{x}_t'' + A_t) = \kappa \cdot \bar{\kappa} \sum_{rt} a_{rt}\mu^{(r)}\bar{\mu}^{(t)} = \kappa \cdot \bar{\kappa} \cos\theta,$$

where θ is the angle between the principal geodesic normals to c and $\bar{c}$ at P. Then (15) or (13) takes the form

$$\kappa^2 + \bar{\kappa}^2 - 2\kappa\bar{\kappa}\cos\theta. \tag{17}$$

Finally, combining (6), (12) and (17), we have

$$\lim_{Q \to P} \frac{2Q\bar{Q}}{(\delta s)^2} = \lim_{Q \to P} \frac{\delta\omega}{\delta s} = \sqrt{\kappa^2 + \bar{\kappa}^2 - 2\kappa\bar{\kappa}\cos\theta}. \tag{18}$$

If, in the definitions (i) and (ii) of § 1, we replace the curve c and the tangent geodesic g by any two tangent curves c and $\bar{c}$, we shall say that the limiting expressions in (1) and (2) define the curvature of c relative to $\bar{c}$ or the *relative curvature* of c and $\bar{c}$. We may now state the following theorem.

Given any two curves c and $\bar{c}$ in V_n tangent at a point P. Let κ and $\bar{\kappa}$ be their respective geodesic curvatures and let θ be the angle between their principal geodesic normals at P. Then the relative curvature λ of c and $\bar{c}$ at P is given by

$$\lambda^2 = \kappa^2 + \bar{\kappa}^2 - 2\kappa\bar{\kappa}\cos\theta.$$

MASSACHUSETTS INSTITUTE OF TECHNOLOGY

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* Bianchi, ibid., p. 364.

SQUARE-PARTITION CONGRUENCES*

BY E. T. BELL

1. *Introduction.* It is evident that the theory of partitions and that of the representation of an integer as a sum of squares must be closely interwoven since both originate in the elliptic theta and modular functions. In seeking the relations thus suggested, we find at the outset some remarkable types of congruences which deserve independent notice on account of their generality. Each congruence is to the odd prime modulus p; the most frequent type concerns the function expressing the number of ways in which an integer is a sum of p, $3p$, p^2, $3p^2$, p^s $(s > 0)$ or rp squares, where r is prime to p, and one of the following: the familiar denumerants of the classical theory of partitions; two new functions depending upon those partitions of an integer in which no part appears more than r times. Of the latter functions those corresponding to $r = 2, 3, 6$ play a central part in the entire theory. The subject is extensive. We shall give a sketch of the methods used sufficient for its systematic development. For the ϑ, q formulas see, e.g., Tannery-Molk, *Fonctions Elliptiques*, and note that we use Jacobi's theta notation (*Werke*, vol. 1, p. 501), so that π is omitted from ϑ_1'.

2. *Fundamental Identities.* In the usual notation $q_j = q_j(q)$,

$$(1)\qquad \begin{aligned} q_0 &= \Pi(1 - q^{2n}), & q_2 &= \Pi(1 + q^m), \\ q_1 &= \Pi(1 + q^{2n}), & q_3 &= \Pi(1 - q^m), \end{aligned}$$

extending to $n = 1, 2, 3, \cdots$, $m = 1, 3, 5, \cdots$, Euler's identities are

$$(2)\qquad q_1q_2q_3 = q_1(\sqrt{q})q_3 = 1, \qquad q_0 = \Sigma(-1)^n q^{3n^2+n},$$

Σ extending to $n = 0, \pm 1, \pm 2, \cdots$. Denote by $A_j(n, r)$ the coefficient† of q^{2n} in q_j^r $(j = 0, 1)$, of q^n in q_j^r $(j = 2, 3)$.

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* Presented to the Society, April 7, 1923.

† The properties of these coefficients have been discussed and a practicable method for their numerical computation given in a paper which will be published in the AMERICAN JOURNAL.

By convention $A_j(0, r) = 1$. By (1) and the first of (2),

$$A_1(n, r) = (-1)^n A_2(n, -r) = A_3(n, -r) \qquad (r \gtrless 0). \tag{3}$$

Henceforth p is an odd prime > 0, r an arbitrary integer > 0. If $s \neq 0, p$, the binomial coefficient $\binom{p}{s}$ is divisible by p, and hence by (1) we have, using Fermat's theorem,

$$q_j^{rp} \equiv q_j^r(q^p) \mod p, \qquad (j = 0, 1, 2, 3), \tag{4}$$

which means that the coefficients of like powers of the parameter q are congruent modulo p. Hence we have by (1) and (3), according as n is or is not prime to p,

$$A_j(n, rp) \equiv 0 \quad \text{or} \quad A_j(n/p, r) \mod p, \tag{5}$$

$$A_1(n, -rp) \equiv 0 \quad \text{or} \quad (-1)^n A_2(n/p, r) \mod p, \tag{6}$$

$$\begin{aligned}(-1)^n A_2(n, -rp) \equiv A_3(n, -rp) &\equiv 0 \\ &\text{or} \quad A_1(n/p, r) \mod p;\end{aligned} \tag{7}$$

and by (4) and the second of (2), according as n is not or is $\frac{1}{2}p(3a^2 + a)(a \gtreqless 0)$,

$$A_0(n, p) \equiv 0, \quad \text{or} \quad (-1)^a \mod p. \tag{8}$$

The summations referring to $n = 0, \pm 1, \pm 2, \cdots$, $m = \pm 1, \pm 3, \pm 5, \cdots$, we have $\vartheta_j = \vartheta_j(q)$,

$$\begin{gathered}\vartheta_0(-q) = \vartheta_3 = \Sigma q^{n^2}, \qquad \vartheta_1'(q^4) = \Sigma(-1|m)mq^{m^2}, \\ \vartheta_2(q^4) = \Sigma q^{m^2},\end{gathered} \tag{9}$$

where $(a|b)$ is the Legendre-Jacobi symbol, $(-1|a) = (-1)^{(a-1)/2}$ for a odd; and

$$\begin{gathered}\vartheta_3 = q_0 q_2^2, \qquad \vartheta_2(q^4) = 2q_0(q^4)q_1^2(q^4)q, \\ \vartheta_1'(q^4) = 2q_0^3(q^4)q.\end{gathered} \tag{10}$$

From the last of these, it follows by (4) that

$$A_0(n, 3p) \equiv 0 \quad \text{or} \quad (-1|a)a \mod p, \tag{11}$$

according as n is not or is $p(a^2 - 1)/8$ where $a > 0$ is odd.

The $A_j(n, r)$ are connected with partitions as follows. If in a given partition of n no part appears more than r times and each of precisely a_j distinct parts occurs exactly j times we call the hypercomplex number $(a_1, a_2, \cdots, a_r)$ the r index of the partition, and denote by $B_n(a_1, a_2, \cdots, a_r)$ or $B_n'(a_1, a_2, \cdots, a_r)$ the total number of partitions of n having this index according as all the parts are not or are restricted to be odd. As mentioned, the cases $r = 2, 3, 6$ are of special importance. For our purpose here it is sufficient by what precedes to consider in these cases $A_j(n, r)$ only when $j = 1, 2$. From (1) we have

$$\begin{aligned} A_1(n, 2) &= \Sigma B_n(a_1, a_2)2^{a_1}, \\ A_1(n, 3) &= \Sigma B_n(a_1, a_2, a_3)3^{a_1+a_2}, \\ A_1(n, 6) &= \Sigma B_n(a_1, \cdots, a_6)2^{a_1+2a_3+a_5}3^{a_1+a_2+a_4+a_5}5^{a_2+a_3+a_4}, \end{aligned}$$

the summations referring to all $(a_1, a_2), \cdots, (a_1, \cdots, a_6)$ for n fixed.* The $A_2(n, r)$ for $r = 2, 3, 6$ are written down from these by accenting B.

Let $P(n)$, $Q(n)$, $R(n)$ denote respectively the total number of partitions of n, the number of partitions of n into odd parts, and the number of partitions of n into distinct odd parts. Then, from (1), (2), we have

$$A_0(n, -1) = P(n), \qquad A_1(n, 1) = Q(n),$$
$$A_2(n, 1) = (-1)^n A_3(n, 1) = R(n).$$

The square functions most frequently occurring are $N(n, r)$, the number of representations of n as a sum of r squares whose roots are $\gtreqless 0$, and $M(n, r)$, the number of representations of n as a sum of r odd squares whose roots are > 0. Obviously $M(n, r) = 0$ if n is not of the form $8k + r$, and

$$(12) \quad \vartheta_3^r(q) = \Sigma q^n N(n, r), \qquad \vartheta_2^r(q^4) = 2^r \Sigma q^{8n+r} M(8n + r, r),$$

where Σ refers to $n = 0, 1, 2, \cdots$, with the convention that $N(0, r) = 1$.

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* The $A_j(n, 2)$ with a generalization have been fully discussed in a paper to appear in the ANNALS OF MATHEMATICS. They are remarkable as introducing for the first time a species of double periodicity into the theory of partitions. The $A_j(n, r)$, $r = 2, 3, 6, 9$, have been specially considered in the paper cited previously; they have many interesting connections with the class number for binary quadratic forms of a negative determinant.

Consider all the representations of n as a sum of r squares of numbers $\gtreqless 0$ of the form $6k+1$, and let $S_0(n, r)$, $S_1(n, r)$ denote the respective numbers of these representations in which an even, an odd number of squares have square roots of the form $12k+7$. Write $S(n, r) = S_0(n, r) - S_1(n, r)$. Then $S(n, r)$ vanishes identically if n is not of the form $24k+r$, and from the second of (2) we have

$$A_0(n, r) = S(24n + r, r), \tag{13}$$

whence it follows by (8) that

$$S(24n + p, p) \equiv 0 \quad \text{or} \quad (-1)^a \quad \text{mod } p, \tag{14}$$

according as n is not or is $\frac{1}{2}p(3a^2 + a)(a \gtreqless 0)$, and by (11)

$$S(24n + 3p, 3p) \equiv 0 \quad \text{or} \quad a(-1|a) \quad \text{mod } p, \tag{15}$$

according as n is not or is $p(a^2-1)/8$, where $a > 0$ is odd.

The congruences in this section appear to be sufficient for the systematic transposition of the classical theory of partitions into congruence relations of the type illustrated in the next. The labor of verifying the congruences numerically may be lightened by observing that $N(n, p)$ is congruent modulo p to twice the total number of representations of n as a sum of p squares with roots all > 0. Similar obvious remarks apply to any of the square functions encountered except those involving only odd squares.

3. *Congruences.* A short selection must suffice. Equating coefficients of like powers of q in $\vartheta_3{}^{rp}q_2{}^{-2rp} = q_0{}^{rp}$ we find

$$\Sigma A_2(s, -2rp)N(2n - s, rp) = A_0(n, rp),$$

the summation referring, as always henceforth unless otherwise noted, to all such $s \geqq 0$ as render the first arguments of the summands positive or zero. Applying (7) we get

$$\Sigma(-1)^s A_1(s, 2r)N(2n - sp, rp) \equiv A_0(n, rp) \quad \text{mod } p, \tag{16}$$

whence by (11) when $r = 3$,

$$\Sigma(-1)^s A_1(s, 6)N(2n - sp, 3p) \equiv 0 \quad \text{or} \quad a(-1|a) \quad \text{mod } p, \tag{17}$$

according as n is not or is $p(a^2 - 1)/8$, where $a > 0$ is odd. The ϑ, q identity being read in the alternative form $\vartheta_3^{rp} = q_0^{rp} q_2^{2rp}$ gives by (5) in the same way

$$N(n, rp) \equiv \Sigma A_0(s, r) A_2(n - 2sp, 2rp) \mod p, \tag{18}$$

from which it follows by (5) that

$$n \not\equiv 0 \mod p: \qquad N(n, rp) \equiv 0 \mod p. \tag{19}$$

From (18) and (5), we have

$$N(np, rp) \equiv \Sigma A_0(s, r) A_2(n - 2s, 2r) \mod p, \tag{20}$$

and hence by (8),

$$N(np, p^2) \equiv \Sigma(-1)^a A_2(n - p(3a^2 + a), 2p) \mod p, \tag{21}$$

where Σ extends to all $a \gtreqless 0$ that make $n \geqq p(3a^2 + a)$. Applying (5) to (21) we have

$$n \not\equiv 0 \mod p: \qquad N(np, p^2) \equiv 0 \mod p; \tag{22}$$

$$N(np^2, p^2) \equiv \Sigma(-1)^a A_2(n - (3a^2 + a), 2) \mod p, \tag{23}$$

where Σ extends to all $a \gtreqless 0$ that make $n \geqq 3a^2 + a$. Again from (20), (11) we find

$$N(np, 3p^2) \equiv \Sigma a(-1|a) A_2(n - \tfrac{1}{4}p(a^2 - 1), 6p) \mod p, \tag{24}$$

where Σ extends to all odd $a > 0$ that make $4n \geqq p(a^2 - 1)$.

Applying (5) to (24), we get

$$n \not\equiv 0 \mod p: \qquad N(np, 3p^2) \equiv 0 \mod p, \tag{25}$$

$$N(np^2, 3p^2) \equiv \Sigma a(-1|a) A_2(n - \tfrac{1}{4}(a^2 - 1), 6) \mod p, \tag{26}$$

where Σ extends to all odd $a > 0$ that make $4n \geqq a^2 - 1$.

Similarly, from the second of (10), we find

$$\Sigma A_1(s, -2r) M(8n - 8s + r, r) = A_0(n, r), \tag{27}$$

$$M(8n + r, r) = \Sigma A_0(s, r) A_1(n - s, 2r). \tag{28}$$

To derive the associated congruences we replace r by rp

and proceed as before. Thus (27) gives

$$(29)\quad \Sigma(-1)^s A_2(s, 2r) M(8n + rp - 8sp, rp) \equiv A_0(n, rp) \mod p;$$

$$(30)\quad \Sigma(-1)^s A_2(s, 6) M(8n + 3p - 8sp, 3p) \equiv 0 \text{ or } a(-1|a) \mod p$$

according as n is not or is $p(a^2 - 1)/8$ where $a > 0$ is odd; while from (28),

$$(31)\quad M(8n + rp, rp) \equiv \Sigma A_0(s, r) A_1(n - sp, 2rp) \mod p;$$

$$(32)\quad n \not\equiv 0 \mod p: \qquad M(8n + rp, rp) \equiv 0 \mod p;$$

$$(33)\quad M(8np + rp, rp) \equiv \Sigma A_0(s, r) A_1(n - s, 2r) \mod p;$$

$$(34)\quad M(8np+p^2, p^2) \equiv \Sigma(-1)^a A_1(n - \tfrac{1}{2}p(3a^2+a), 2p) \mod p,$$

where Σ extends to all $a \gtreqless 0$ that make $2n \geqq p(3a^2 + a)$;

$$(35)\quad M(8np^2+p^2, p^2) \equiv \Sigma(-1)^a A_1(n - \tfrac{1}{2}(3a^2+a), 2) \mod p,$$

where Σ extends to all $a \gtreqless 0$ that make $2n \geqq 3a^2 + a$;

$$(36)\quad n \not\equiv 0 \mod p: \qquad M(8np + p^2, p^2) \equiv 0 \mod p;$$

$$(37)\quad M(8np + 3p^2, 3p^2) \equiv \Sigma a(-1|a) A_1(n - \tfrac{1}{8}p(a^2 - 1), 6p) \mod p;$$

$$(38)\quad n \not\equiv 0 \mod p: \qquad M(8np + 3p^2, 3p^2) \equiv 0 \mod p;$$

$$(39)\quad M(8np^2 + 3p^2, 3p^2) \equiv \Sigma a(-1|a) A_1(n - \tfrac{1}{8}(a^2 - 1), 6) \mod p,$$

the summations in (37), (39) extending to all odd $a > 0$ making the first arguments of $A_1 \geqq 0$. Putting $r = 1$ in (29) and applying (13), (14) we find

$$(40)\quad \Sigma(-1)^s A_2(s, 2) M(8n + p - 8sp, p) \equiv 0 \text{ or } (-1)^a \mod p$$

according as n is not or is $\tfrac{1}{2}p(3a^2 + a)(a \gtreqless 0)$. Similarly, from (16), under the same conditions, we have

$$(41)\quad \Sigma(-1)^s A_1(s, 2) N(n - sp, p) \equiv 0 \text{ or } (-1)^a \mod p.$$

The number of congruences obtainable in this way is practically unlimited. Thus the memoir of Jacobi * on infinite series whose exponents are contained simultaneously in two different quadratic forms alone furnishes an inexhaustible supply, and the modular equations in elliptic functions give many more. Quadratic forms other than simple sums of squares appear in this connection. For example consider all the representations of n in the form

$$a_1^2 + a_2^2 + \cdots + a_r^2 + 3(b_1^2 + b_2^2 + \cdots + b_r^2),$$

in which $a_j,\ b_j \lessgtr 0$ and $a_j \equiv 1 \mod 6,\ b_j \equiv 1 \mod 4$ $(j = 1, 2, \cdots, r)$. Let $T_0(n, r)$ denote the total number of these representations in which an even number of the b_j are of the form $8k + 5$, and $T_1(n, r)$ the total number in which an odd number of the b_j are of the form $8k + 5$. Write

$$T(n, r) = T_0(n, r) - T_1(n, r).$$

Then Jacobi's result (*Werke*, vol. 2, p. 285)

$$q^4 q_0^2(q^{24}) = \Sigma(-1)^k q^{(6i+1)^2+3(4k+1)^2},$$

where Σ refers to $i, k = -\infty$ to $+\infty$, gives

$$A_0(n, 2r) = T(48n + 4r, r),$$

from which we find by successive applications of (5), for $s \geqq 0$,

$$(42)\quad n \equiv 0 \mod p: \quad T(48np^s + 4rp^{s+1}, rp^{s+1}) \equiv 0 \mod p,$$

$$(43)\quad T(48np^{s+1} + 4rp^{s+1}, rp^{s+1}) \equiv A_0(n, 2r) \mod p,$$

$$(44)\quad T(48np^{s+1} + 4p^{s+1}, p^{s+1}) \equiv A_0(n, 2) \mod p.$$

Considerations of space preclude the giving of further examples.

THE UNIVERSITY OF WASHINGTON

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* *Werke*, vol. 2, pp. 219–288; CRELLE'S JOURNAL, vol. 37, pp. 61–94, 221–254.

DETERMINATION OF ALL SYSTEMS OF ∞^4 CURVES IN SPACE IN WHICH THE SUM OF THE ANGLES OF EVERY TRIANGLE IS TWO RIGHT ANGLES*

BY JESSE DOUGLAS

1. *Introduction.* Consider the curves which intersect an arbitrarily chosen system of ∞^1 curves in the plane under a fixed angle α. If α is varied, a system of ∞^2 curves is obtained, termed an *isogonal* family. Isogonal families are characterized by differential equations of the form

$$y'' = (T_x + y'T_y)(1 + y'^2), \tag{1}$$

where T is any function of x and y.

It is easy to prove synthetically that in all isogonal families, and in no other systems of ∞^2 curves in the plane, the sum of the angles of the triangle formed by any three of the curves is equal to π.†

A *natural* family of curves in any space is one obtainable as the system of extremals of a calculus of variations problem of the form

$$\int F ds = \text{minimum}, \tag{2}$$

where F is any point function.‡ In the plane, F is a function of x and y, and the Euler-Lagrange equation of (2) is

$$y'' = (L_y - y'L_x)(1 + y'^2), \tag{3}$$

where $L = \log F$.

Since the family formed by the ∞^2 straight lines of the plane is both isogonal and natural, and since each of these characters is invariant under conformal transformation, every curve family conformally equivalent to the straight lines must be both isogonal and natural.

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* Presented to the Society, April 28, 1923.

† G. Scheffers, *Isogonalkurven, Äquitangentialkurven und komplexe Zahlen*, MATHEMATISCHE ANNALEN, vol. 60 (1905), p. 504.

‡ See E. Kasner, PRINCETON COLLOQUIUM LECTURES (1912), pp. 34–37.

Conversely, for a curve family which is at once isogonal and natural, we have by identification of (1) with (3)

$$T_x = L_y, \qquad T_y = -L_x. \tag{4}$$

These equations imply that $L - iT$ and consequently e^{L-iT} are analytic functions of $x + iy$. Since the conformal transformation

$$x_1 + iy_1 = e^{L-iT},$$

transforms $e^L ds$ into ds_1, it converts the extremals of $\int F ds = \int e^L ds =$ minimum into those of $\int ds_1 =$ minimum, that is, into the straight lines.

The conditions (4) can be satisfied only when T is Laplacian. It follows that the property of having the angle sum in each triangle equal to π is, in the plane, not restricted to the curve families derivable by conformal transformation from the straight lines. In contrast with this fact, it is the object of the present paper to prove the following theorem.

THEOREM. *If a system of ∞^4 curves in space is such that in each of its triangles the sum of the angles is equal to π, then it is either the system formed by the ∞^4 straight lines of space, or an image of that system by a conformal transformation of space, namely the ∞^4 circles through a fixed point.*

It is presumed in the statement of this theorem that the only systems of ∞^4 curves in space taken into consideration are those that can be defined by a system of two differential equations of the form

$$y'' = F(x, y, z, p, q), \qquad z'' = G(x, y, z, p, q), \tag{5}$$

where p, q represent y', z' respectively, and where F and G are analytic functions of their five arguments. We select a region of the x, y, z, p, q continuum, within which each of these functions has a branch which is uniform and regular, and we restrict ourselves in the calculations that follow to this region and to these branches.

2. *A Triangle with One Infinitesimal Angle.* We shall use the symbol $\mathfrak{F}$ to denote any quadruply infinite family of curves defined by equations of the form (5). Under the restrictions just stated, there will pass through each point and in each direction a unique curve of $\mathfrak{F}$.

Choose any point 0—denote its coordinates by x_0, y_0, z_0—and let $1 : p_0 : q_0$ and $1 : p_0 + \delta p_0 : q_0 + \delta q_0$ define two directions through 0 infinitely near to one another. These determine in the family $\mathfrak{F}$ two consecutive curves C_1 and C_2. Let the equations of C_1 be

$$Y = y(X), \qquad Z = z(X); \tag{6}$$

then those of C_2 will be

$$\begin{aligned} Y &= y(X) + \delta p_0 \eta_1(X) + \delta q_0 \eta_2(X), \\ Z &= z(X) + \delta p_0 \zeta_1(X) + \delta q_0 \zeta_2(X), \end{aligned} \tag{7}$$

where η_1, ζ_1 and η_2, ζ_2 obey the equations of variation of the system (5)

$$\begin{aligned} \eta'' &= F_p \eta' + F_q \zeta' + F_y \eta + F_z \zeta, \\ \zeta'' &= G_p \eta' + G_q \zeta' + G_y \eta + G_z \zeta, \end{aligned} \tag{8}$$

and are completely determined by the additional data

$$(9_1) \quad \eta_1(x_0) = 0, \qquad \zeta_1(x_0) = 0, \quad \eta_1'(x_0) = 1, \qquad \zeta_1'(x_0) = 0,$$
$$(9_2) \quad \eta_2(x_0) = 0, \qquad \zeta_2(x_0) = 0, \quad \eta_2'(x_0) = 0, \qquad \zeta_2'(x_0) = 1.$$

It is to be understood that in the coefficients of (8), which are originally functions of x, y, z, p, q, we are to substitute by means of (6)

$$x = X, \quad y = y(X), \quad z = z(X), \quad p = y'(X), \quad q = z'(X),$$

so that these coefficients become functions only of X, the abscissa along C_1.

On C_1 let any point $1(x, y, z)$ other than 0 be selected, and let $2(x + \delta x,\ y + \delta y,\ z + \delta z)$ be an infinitely near point on C_2. Through 1 and 2 there passes a unique curve of the family $\mathfrak{F}$; designate it as $\bar{C}$, and denote by $1 : \bar{p} : \bar{q}$ and $1 : \bar{p} + \delta \bar{p} : \bar{q} + \delta \bar{q}$

its directions at 1 and 2 respectively. Then

$$\delta y = \bar{p}\delta x, \qquad \delta z = \bar{q}\delta x, \tag{10}$$

and

$$\delta\bar{p} = \bar{F}\delta x, \qquad \delta\bar{q} = \bar{G}\delta x, \tag{11}$$

where the bars over F and G indicate that these functions are to be formed for the arguments x, y, z, $\bar{p}$, $\bar{q}$.

Allow $1 : p : q$ and $1 : p + \delta p : q + \delta q$ to represent respectively the direction of C_1 at 1 and of C_2 at 2; then by differentiation of (6) and (7), substitution of the coordinates of 1 and 2, and use of (5),

$$p = y'(x), \qquad q = z'(x),$$

and

$$\begin{cases} \delta p = F\delta x + \delta p_0\eta_1' + \delta q_0\eta_2', \\ \delta q = G\delta x + \delta p_0\zeta_1' + \delta q_0\zeta_2', \end{cases} \tag{12}$$

where the arguments in F, G are the x, y, z, p, q of the point 1 and curve C_1, and in η_1', ζ_1', η_2', ζ_2' the argument is x.

The fact that the curve C_2 or (7) contains the point 2 gives, with the use of (10),

$$\begin{aligned} (\bar{p} - p)\delta x &= \delta p_0\eta_1 + \delta q_0\eta_2, \\ (\bar{q} - q)\delta x &= \delta p_0\zeta_1 + \delta q_0\zeta_2; \end{aligned}$$

therefore

$$\begin{cases} \delta x : \delta p_0 : \delta q_0 = \eta_1\zeta_2 - \eta_2\zeta_1 : [(\bar{p} - p)\zeta_2 - (\bar{q} - q)\eta_2] \\ \qquad\qquad\qquad\qquad : [-(\bar{p} - p)\zeta_1 + (\bar{q} - q)\eta_1]. \end{cases} \tag{13}$$

In the curvilinear triangle 012 or $C_1C_2\bar{C}$ let $\delta\omega$ denote the interior angle at 0, θ the interior angle at 1, and $\theta + \delta\theta$ the exterior angle at 2. Then the condition for an angle sum equal to two right angles is

$$\delta\theta = \delta\omega. \tag{14}$$

Now

$$\cos\theta = \frac{1 + p\bar{p} + q\bar{q}}{\sqrt{1 + p^2 + q^2}\sqrt{1 + \bar{p}^2 + \bar{q}^2}}.$$

Therefore

$$
\begin{aligned}
-\sin\theta\,\delta\theta = {} & \frac{1+p\bar{p}+q\bar{q}}{\sqrt{1+p^2+q^2}\,\sqrt{1+\bar{p}^2+\bar{q}^2}} \\
& \times\left\{\frac{\bar{p}\delta p+\bar{q}\delta q+p\delta\bar{p}+q\delta\bar{q}}{1+p\bar{p}+q\bar{q}}\right. \\
& \qquad \left. -\frac{p\delta p+q\delta q}{1+p^2+q^2}-\frac{\bar{p}\delta\bar{p}+\bar{q}\delta\bar{q}}{1+\bar{p}^2+\bar{q}^2}\right\},
\end{aligned}
\tag{15}
$$

where the value of sin θ is

$$
\frac{\sqrt{(1+q^2)(\bar{p}-p)^2-2pq(\bar{p}-p)(\bar{q}-q)+(1+p^2)(\bar{q}-q)^2}}{\sqrt{1+p^2+q^2}\,\sqrt{1+\bar{p}^2+\bar{q}^2}}\,. \tag{16}
$$

Besides,

$$
\delta\omega=\frac{\sqrt{(1+q_0{}^2)\delta p_0{}^2-2p_0q_0\delta p_0\delta q_0+(1+p_0{}^2)\delta q_0{}^2}}{1+p_0{}^2+q_0{}^2}\,. \tag{17}
$$

Combining the equations (14) to (17), we have

$$
\begin{aligned}
-(1+p\bar{p}+q\bar{q}) & \left\{\frac{\bar{p}\delta p+\bar{q}\delta q+p\delta\bar{p}+q\delta\bar{q}}{1+p\bar{p}+q\bar{q}}\right. \\
& \qquad \left. -\frac{p\delta p+q\delta q}{1+p^2+q^2}-\frac{\bar{p}\delta\bar{p}+\bar{q}\delta\bar{q}}{1+\bar{p}^2+\bar{q}^2}\right\} \\
= {} & \frac{1}{1+p_0{}^2+q_0{}^2}\sqrt{(1+q_0{}^2)\delta p_0{}^2-2p_0q_0\delta p_0\delta q_0+(1+p_0{}^2)\delta q_0{}^2} \\
& \times\sqrt{(1+q^2)(\bar{p}-p)^2-2pq(\bar{p}-p)(\bar{q}-q)+(1+p^2)(\bar{q}-q)^2}.
\end{aligned}
\tag{18}
$$

By means of the substitutions indicated by (11) and (12), the use of (13), the introduction of

$$
\left\{
\begin{aligned}
\phi &= \frac{(1+q^2)F-pqG}{1+p^2+q^2}, \\
\psi &= \frac{-pqF+(1+p^2)G}{1+p^2+q^2},
\end{aligned}
\right.
\tag{19}
$$

also of symbols ω_i to represent the two-rowed determinants in the matrix

$$
\begin{Vmatrix}
\eta_1 & \zeta_1 & \eta_1{}' & \zeta_1{}' \\
\eta_2 & \zeta_2 & \eta_2{}' & \zeta_2{}'
\end{Vmatrix}
$$

as follows,

$$\omega_1 = (\eta\zeta), \qquad \omega_2 = (\eta\eta'), \qquad \omega_3 = (\eta\zeta'),$$
$$\omega_4 = (\zeta\eta'), \qquad \omega_5 = (\zeta'\zeta), \qquad \omega_6 = (\eta'\zeta'),$$

and of

$$\begin{aligned} \Omega_1 &= (1+q^2)\omega_4 + pq\omega_5, \\ \Omega_2 &= -(1+q^2)\omega_2 + pq\omega_3 - pq\omega_4 - (1+p^2)\omega_5, \\ \Omega_3 &= pq\omega_2 - (1+p^2)\omega_3, \end{aligned}$$

finally of

$$(20)\quad \begin{cases} I = (1+p_0^2)\eta_1^2 + 2p_0q_0\eta_1\eta_2 + (1+q_0^2)\eta_2^2, \\ II = (1+p_0^2)\eta_1\zeta_1 + p_0q_0(\eta_1\zeta_2 + \eta_2\zeta_1) + (1+q_0^2)\eta_2\zeta_2, \\ III = (1+p_0^2)\zeta_1^2 + 2p_0q_0\zeta_1\zeta_2 + (1+q_0^2)\zeta_2^2, \end{cases}$$

the condition (18) reduces to

$$\begin{aligned} &(1+p^2+q^2)\omega_1\{(\bar\phi-\phi)(\bar p-p) + (\bar\psi-\psi)(\bar q-q)\} \\ &\quad + \Omega_1(\bar p-p)^2 + \Omega_2(\bar p-p)(\bar q-q) + \Omega_3(\bar q-q)^2 \\ (21)\qquad &= \frac{1+p^2+q^2}{1+p_0^2+q_0^2}\sqrt{III(\bar p-p)^2 - 2II(\bar p-p)(\bar q-q) + I(\bar q-q)^2} \\ &\quad \times \sqrt{(1+q^2)(\bar p-p)^2 - 2pq(\bar p-p)(\bar q-q) + (1+p^2)(\bar q-q)^2}. \end{aligned}$$

It is to be observed that ω_1, Ω_1, Ω_2, Ω_3, and I, II, III are independent of $\bar p$, $\bar q$.

Since the right member of (21) and the second line of the left member are homogeneous functions of the second degree in $\bar p - p$, $\bar q - q$, the same must be true of

$$(22)\qquad (\bar\phi-\phi)(\bar p-p) + (\bar\psi-\psi)(\bar q-q).$$

By Taylor's theorem,

$$\begin{aligned} \bar\phi-\phi &= \phi_p(\bar p-p) + \phi_q(\bar q-q) \\ &\quad + \tfrac{1}{2}\{\phi_{pp}(\bar p-p)^2 + 2\phi_{pq}(\bar p-p)(\bar q-q) + \phi_{qq}(\bar q-q)^2\} + \cdots, \\ \bar\psi-\psi &= \psi_p(\bar p-p) + \psi_q(\bar q-q) \\ &\quad + \tfrac{1}{2}\{\psi_{pp}(\bar p-p)^2 + 2\psi_{pq}(\bar p-p)(\bar q-q) + \psi_{qq}(\bar q-q)^2\} + \cdots. \end{aligned}$$

Necessary conditions that (22) be homogeneous of the second degree in $\bar{p} - p$, $\bar{q} - q$ are seen to be

$$\phi_{pp} = 0, \quad 2\,\phi_{pq} + \psi_{pp} = 0, \quad \phi_{qq} + 2\,\psi_{pq} = 0, \quad \psi_{qq} = 0,$$

a system of partial differential equations whose solution is

$$\begin{aligned} \phi &= \beta q^2 - \gamma pq + \lambda p + \mu q + \nu, \\ \psi &= \gamma p^2 - \beta pq + \lambda' p + \mu' q + \nu'; \end{aligned} \tag{23}$$

and these forms of ϕ, ψ are seen to be also sufficient for the condition in question. β, γ, λ, etc. are functions only of x, y, z.

The left member of (21) now becomes a rational entire function of $\bar{p} - p$, $\bar{q} - q$; in order that the same be true of the right member we must have

$$\frac{I}{1 + p^2} = \frac{II}{pq} = \frac{III}{1 + q^2}. \tag{24}$$

By means of (8), (9), and (20), I, II, III can be expressed as power series in $t = x - x_0$; we find

$$\begin{aligned} I &= (1 + p_0^2)t^2 + \{(1 + p_0^2)F_p{}^0 + p_0q_0F_q{}^0\}t^3 + \cdots, \\ II &= p_0q_0t^2 + \{(1 + p_0^2)G_p{}^0 + p_0q_0(F_p{}^0 + G_q{}^0) \\ &\qquad + (1 + q_0^2)F_q{}^0\}t^3 + \cdots, \\ III &= (1 + q_0^2)t^2 + \{p_0q_0G_p{}^0 + (1 + q_0^2)G_q{}^0\}t^3 + \cdots. \end{aligned} \tag{25}$$

Furthermore,

$$\begin{aligned} 1 + p^2 &= 1 + p_0{}^2 + 2p_0F_0t + \cdots, \\ pq &= p_0q_0 + (p_0G_0 + q_0F_0)t + \cdots, \\ 1 + q^2 &= 1 + q_0{}^2 + 2q_0G_0t + \cdots. \end{aligned} \tag{26}$$

When, after dividing the members of (25) by the corresponding ones of (26), we equate coefficients of t^3, we obtain

$$\begin{aligned} &\frac{(1 + p^2)F_p + pqF_q - 2pF}{1 + p^2} \\ &= \frac{(1+p^2)G_p + pq(F_p + G_q) + (1+q^2)F_q - (pG + qF)}{2pq} \\ &= \frac{pqG_p + (1 + q^2)G_q - 2qG}{1 + q^2}, \end{aligned} \tag{27}$$

where the subscript and superscript 0 has been dropped with evident justification.

Writing

$$F = (1 + p^2)\phi + pq\psi,\qquad G = pq\phi + (1 + q^2)\psi, \tag{28}$$

obtained by reversion of (19), and then applying the values of ϕ and ψ given by (23), we find that (27) imposes the conditions

$$\nu = \beta, \qquad \nu' = \gamma, \qquad \lambda' = 0, \qquad \mu = 0, \qquad \lambda = \mu',$$

which by (23) and (28) reduce F and G to

$$F = (\beta - \alpha p)(1 + p^2 + q^2),\qquad G = (\gamma - \alpha q)(1 + p^2 + q^2), \tag{29}$$

where we have replaced $\lambda = \mu'$ by $-\alpha$.

3. *The* ∞^3 *Surfaces* Σ. Relative to the ∞^4 straight lines of space there is a family of ∞^3 surfaces, namely the planes, of which the straight lines are the mutual intersections. Not every system of ∞^4 curves in space has a so related family of ∞^3 surfaces. We have found the necessary and sufficient conditions to be

$$\begin{aligned} &F_q(F_p + G_q) + 4F_z - 2F_q' = 0,\\ &F_p^2 - G_q^2 + 4(F_y - G_z) - 2(F_p' - G_q') = 0,\\ &G_p(F_p + G_q) + 4G_y - 2G_p' = 0. \end{aligned} \tag{30}$$

The accent denotes the operator

$$\frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + q\frac{\partial}{\partial z} + F\frac{\partial}{\partial p} + G\frac{\partial}{\partial q}\cdot$$

Let A be any point of space, through which d_1 and d_2 are any two directions, and let Γ_1 and Γ_2 denote the curves of the family $\mathfrak{F}$ which pass through A in these directions respectively. Choose arbitrarily a point B on Γ_1 and a point C on Γ_2. There is a unique curve BC of $\mathfrak{F}$ which passes through B and C.

On BC let D be any point. Suppose the curve determined by A and D to have at A the direction d.

Then under the hypothesis of an angle sum in every triangle equal to π, the direction d must belong to the flat pencil determined by d_1 and d_2. For by applying this hypothesis to the triangles ABD, ACD, ABC, it may be deduced that

$$\measuredangle d_1 d + \measuredangle d d_2 = \measuredangle d_1 d_2,$$

which implies that d is coplanar with d_1 and d_2.

The curves of $\mathfrak{F}$ which radiate from A in the directions of the pencil determined by d_1 and d_2 form a surface Σ. By the above, Σ contains AD; therefore it contains D; therefore it contains the curve BC, since D was an arbitrary point of BC.

Now B and C were arbitrary points on Γ_1 and Γ_2 respectively. It follows that through each pair of intersecting curves of $\mathfrak{F}$ there passes a surface Σ which carries ∞^2 curves of $\mathfrak{F}$. It is easy to see how this implies that the ∞^4 curves of $\mathfrak{F}$ are the mutual intersections of ∞^3 surfaces.

The functions F and G, so far reduced to the forms (29), must therefore obey (30). The imposition of (30) restricts F and G further, namely to the forms

$$\left\{\begin{aligned} F &= (L_y - pL_x)(1 + p^2 + q^2), \\ G &= (L_z - qL_x)(1 + p^2 + q^2), \end{aligned}\right. \tag{31}$$

where either

$$L = f(x^2 + y^2 + z^2), \tag{32a}$$

or

$$L = g(y). \tag{32b}$$

In the reduction of L, use is made, in general, of a transformation of the axes.

In other words, the differential equations of $\mathfrak{F}$ are now reduced to one or the other of the forms

$$\begin{aligned} y'' &= \rho(y - y'x)(1 + y'^2 + z'^2), \\ z'' &= \rho(z - z'x)(1 + y'^2 + z'^2), \qquad \rho = 2f'(x^2 + y^2 + z^2); \end{aligned} \tag{33a}$$

$$\begin{aligned} y'' &= g'(y)(1 + y'^2 + z'^2), \\ z'' &= 0. \end{aligned} \tag{33b}$$

4. *Final Conditions.* We consider first (33*a*). Its two equations can be combined so as to give

$$\begin{vmatrix} x & y & z \\ 1 & y' & z' \\ 0 & y'' & z'' \end{vmatrix} = 0,$$

whose general integral is $Ax + By + Cz = 0$, where A, B, C are arbitrary constants. It follows that each curve of $\mathfrak{F}$ lies in a plane that passes through the origin O. The ∞^2 curves of $\mathfrak{F}$ carried by each surface Σ are therefore the sections of Σ by the planes through O (provided that Σ is not itself a plane through O *).

We next observe that the equations (31) are the Euler-Lagrange equations for $\int e^L ds =$ minimum. But if a curve on a surface Σ is an extremal of $\int e^L ds$ relative to space, it is a fortiori an extremal relative to Σ. Thus the ∞^2 curves on Σ form, according to the definition in § 1, a *natural* family on Σ.

Imagine Σ to be represented conformally on the plane. Then this natural family on Σ goes over into a natural family in the plane. For conformal transformation multiplies ds by a point function; therefore the extremals of an integral of the form $\int$ *point function* $\cdot\, ds$ are transformed into the extremals of an integral of the same form.

Moreover, in this family of ∞^2 curves in the plane the sum of the angles of every triangle is equal to π; the family is therefore *isogonal* as well as natural. By § 1, it is therefore convertible into the ∞^2 straight lines by a conformal transformation of the plane. In this way, it is possible to convert the original system of ∞^2 curves on Σ into the straight lines of the plane by a conformal representation of Σ on the plane.

We have therefore to deal with the following problem: If O is a point of space, and Σ a surface not a plane through O, what is implied by the circumstance that Σ admits of a conformal representation on the plane in which its ∞^2 sections by the planes through O go over into the ∞^2 straight lines of the plane?

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* Each curve of $\mathfrak{F}$ belongs to ∞^1 surfaces Σ. These cannot all be planes through O unless the curve is a straight line through O, a case which may be laid aside without affecting the validity of our ultimate conclusion.

We say that Σ must be either a plane, or else a sphere passing through O.

For consider any minimal line of the plane. In the conformal representation it must correspond to a curve on Σ which (1) is a plane curve, (2) has zero length. A plane curve of zero length is necessarily a minimal straight line. The surface Σ therefore carries two systems of minimal straight lines.

A surface doubly covered by minimal straight lines is either a plane or a sphere. If Σ is a plane, the condition in question is satisfied without further restriction.

But if Σ is a sphere, it must, in addition, pass through O. For the angle sum in a triangle formed by three circles on a sphere, whose planes intersect in a point O, is greater than π if O is interior to the sphere, and less than π if O is exterior to the sphere. Moreover, the condition Σ is a sphere which passes through O, is a sufficient one, for then the stereographic projection of Σ from O as pole is a conformal representation of Σ on the plane which converts the sections of Σ by the planes through O into straight lines.

It follows that each curve of $\mathfrak{F}$ must be either a straight line, or a circle through O. Since each of the three curve families $\mathfrak{F}$, the straight lines of space, the circles through O, is an irreducible analytic manifold of four dimensions, $\mathfrak{F}$ must be identical either with the ∞^4 straight lines of space, or with the ∞^4 circles through O.

The case (33*b*) can by a similar argument be proved to lead only to the straight lines.

Columbia University

VOLUME III OF LIE'S MEMOIRS

Sophus Lie's Gesammelte Abhandlungen (*Samlede Avhandlinger*). Edited by Friedrich Engel and Poul Heegaard. Volume III. *Abhandlungen zur Theorie der Differentialgleichungen*, erste Abteilung (Avhandlinger til Differentialligningernes Teori, første Avdeling), edited by Friedrich Engel. Leipzig, B. G. Teubner, and Kristiania, H. Aschehoug and Co., 1922. xvi + 789 pages.

Twenty-three years after the death of Sophus Lie appears the first volume to be printed of his collected memoirs. It is not that nothing has been done in the meantime towards making his work more readily available. A consideration of the matter was taken up soon after his death but dropped owing to the difficulties in the way of printing so large a collection as his memoirs will make. An early and unsuccessful effort to launch the enterprise was made by the officers of Videnskapsselskapet i Kristiania; but plans did not take a definite form till 1912; then through the Mathematisch-physische Klasse der Leipziger Akademie and the publishing firm of B. G. Teubner steps were taken to launch the project. Teubner presented a plan for raising money by subscription to cover a part of the cost of the work and a little later invitations to subscribe were sent out. The responses were at first not encouraging; from Norway, the homeland of Lie, only three subscriptions were obtained in response to the first invitations.

In these circumstances, Engel, who was pressing the undertaking, resorted to an unusual means. He asked the help of the daily press of Norway. On March 9, 1913, the newspaper TIDENS TEGN of Christiania carried a short article by Engel with the title *Sophus Lies samlede Afhandlinger* in which was emphasized the failure of Lie's homeland to respond with assistance in the work of printing his collected memoirs. This attracted the attention of the editor and he took up the campaign: two important results came from this, namely, a list of subscriptions from Norway to support the undertaking and an appropriation by the Storthing to assist in the work. By June the amount of support received and promised was sufficient to cause Teubner to announce that the work could be undertaken; and in November the memoirs for the first volume were sent to the printer, the notes and supplementary matter to be supplied later.

The Great War so interfered with the undertaking that it could not be continued, and by the close of the war circumstances were so altered that the work could not proceed on the basis of the original subscriptions and understandings and new means for continuing the work had to be sought. Up to this time the work had been under the charge of Engel as editor. But it now became apparent that the publication of the memoirs would have to become a Norwegian undertaking. Accordingly, Poul Heegaard became associated with Engel as an editor. The printing of the work became an enterprise not of the publishers but of the societies which support them in this undertaking. Under such circumstances the third volume of the series, but the first one to be printed, has now been put into our hands. "The printing of further volumes will be carried through gradually as the necessary means are procured; more I cannot say about it," says Engel, "because the cost of printing continues to mount incessantly."

The general plan contemplates the publication of Lie's memoirs in six volumes while a seventh volume is to be devoted to the principal works found among Lie's literary remains. Volumes I and II are to contain the memoirs on geometry; volumes III and IV, those on differential equations; and volumes V and VI, those on transformation groups. Naturally these three divisions are not rigorously separated one from another; in fact, in most of the memoirs in a given part appears matter relating also to the other two parts. In each part the memoirs are arranged chronologically; for the most part, the material in the first of two related volumes is that which was first published at Christiania, while the material in the second is taken principally from the MATHEMATISCHE ANNALEN and the publications of the Leipzig Akademie. In this way an arrangement is effected by which two articles never appear in the same volume one of which is a reworking of the other. Every memoir already has its definite place assigned to it in the completed work and a definite number in the volume to which it belongs is given to it. In this way it becomes possible to give cross references from volume to volume without any possibility of confusion. In the (unexpected) event of the discovery of an article at first overlooked, it will be assigned to its chronological position, with a number followed by a star, so as not to disturb the numbering of the other articles.

To facilitate the examination of references by means of the collected works and without the use of the articles in their original place of publication, the exact paging of the original articles is indicated by means of numbers inserted in the lines at the appropriate places. This is a feature of convenience which is to be commended. By means of it the paging of the original is easily determined from the collected memoirs.

The publication is started with the third volume because the letters of Lie furnish such an abundant material for the explanation of the memoirs in it; and this material has been used freely. The letters to A. Mayer have been particularly useful in this respect. Besides the material gathered from the letters there is a rich copiousness of notes and explanations by Engel. These have been prepared with the purpose of making the memoirs in this volume more readily understood by the reader. The editor anticipates that the reader may sometimes find these too extensive; but, in case this evil is found to exist in them, he has thought that it is much less than the possible opposite evil of notes which are not sufficiently full. The theory of partial differential equations of the first order and of Pfaff's problem was greatly modified by the researches of Lie. Concerning the elucidation of these researches in the notes Engel speaks as follows in his preface: "Ich habe nun versucht auseinanderzusetzen, wie Lie etwa ursprünglich zu seinen Sätzen gelangt ist, namentlich habe ich mich bemüht, für die Theorie der partiellen Differentialgleichungen 1.O. nach Möglichkeit den Standpunkt wiederherzustellen, auf dem Lie war, als er die ersten Abhandlungen des Bandes (Nr. I–IV) veröffentlichte. Obwohl ich mir bewusst war, mir damit eine äusserst schwierige, fast unlösbare Aufgabe gestellt zu haben, habe ich doch den Versuch gewagt, und ich hoffe, in den Anmerkungen zu Abhandlung I [covering 38 pages] wenigstens zum Teil das Richtige getroffen zu haben; vollständig befriedigt bin ich auch selber noch nicht, obwohl ich gerade diese Anmerkungen ein halbesdutzendmal umgearbeit habe. Sicherer bin ich meiner Sache beim Pfaffschen Probleme; ich darf wohl behaupten, dass meine Anmerkungen zu Abhandlung XI [16 pages] den ursprünglichen Gedankengang Lies im wesentlichen richtig wiedergeben."

The memoirs of Lie reprinted in the present volume cover 562 of the total 805 pages. But the notes and supplementary matter are printed much more compactly than the memoirs themselves, so that the former make up distinctly more than one third of the whole volume. This additional material has been prepared with great care and with the convenience of the reader always in mind. The articles originally printed in Norwegian appear now in a German translation. The few notes in French are printed in that language except in the case of notes which Lie himself had published both in French and in German, these latter being reprinted in German with a reference also to the place of publication in French. All of Lie's earlier work placed by the editors under differential equations is therefore readily available to all who read German (the material in the French notes being only brief abstracts); and this is placed before the reader in convenient form with an ample richness of explanatory material.

In his preface Engel insists upon certain qualities of Lie's work—qualities which partly explain the copiousness of the notes which he has inserted in the present volume. The ideas and points of view which Lie possessed and never adequately made known to others were irretrievably lost with his death. In his printed works there are many suggestions which were not followed up by him. These are scattered, and are largely inaccessible. When his complete works are before us in convenient form, these fragmentary suggestions can be seen in their totality and they will throw light one upon another. In this way Lie will become his own interpreter. This will be particularly true when the material in the letters is brought to bear upon the researches to which its parts refer. It seems important therefore to bring the scattered suggestions of Lie together in connection with the memoirs to which they may properly be attached.

The editor's careful work in preparing this volume for the printer and in seeing it through the press has evidently been for him a labor of love; in performing it he has rendered a great service to all those who will have occasion to use the memoirs of Lie which are reprinted in this volume. At present the intention is that the fifth volume shall follow the third; but no definite promises are being made owing to the difficulties which we have already named. The memoirs for the fifth volume are now ready for the printer; but the notes for it will be prepared while the volume is in course of being printed. It is impossible yet to tell what the extent of these notes will be; but the editor states that they will be far less extensive than those in the present volume. All mathematical libraries will of course procure copies of the volume now printed. It is to be hoped also that every individual who can utilize a private copy of the volume effectively will purchase one so as to facilitate the publication of the next volumes.

R. D. Carmichael

BOLZANO ON PARADOXES

Paradoxien des Unendlichen. By Bernard Bolzano, mit Anmerkungen versehen von Hans Hahn. Leipzig, Meiner, 1921. 12 + 157 pp.

This is a new edition, with notes by Hans Hahn, of the book first published in 1851, three years after Bolzano's death. It appears to have remained for many years almost unknown; for L. Couturat makes no mention of it in his book *De L'Infini Mathématique* (1896) except to state in a note that he came across it when his own book had already been entirely printed. Stolz includes a consideration of this book in an article (MATHEMATISCHE ANNALEN, 1881, p. 255) giving an estimate of Bolzano's work and influence, and states that several years before Cauchy published his lectures on the calculus, Bolzano had developed the fundamental concepts of the calculus which in many respects agree with those of Cauchy, but which in important respects are more complete. Furthermore Hankel attributes to Bolzano priority over Cauchy of the proper conception of infinite series.

The reviewer does not think it fair to criticize severely from the standpoint of present standards of rigor a book written about seventy-five years ago; but in view of the high estimates put on Bolzano's work, it does seem desirable to mention a few of the errors.

Bolzano is concerned with a discussion of Gergonne's solution of the series

$$a - a + a - a + \cdots. \tag{1}$$

The solution is as follows. Let x be the value of the series; then

$$x = a - a + a - a + \cdots = a - (a - a + a - a + \cdots),$$

that is,

$$x = a - x, \qquad \text{and} \qquad x = \frac{a}{2}.$$

Bolzano criticizes this solution in two ways. In the first place he says that the series in parentheses is not identical with the series (1), because regarded as a set of terms it lacks the first term a. This assertion of Bolzano's is of course erroneous; in fact if it be granted that the series (1) have a value at all, then Gergonne's solution is correct. In the second place Bolzano objects altogether to attaching a value to the series (1). While this objection is quite legitimate, the *grounds* for the objection are not. The series can have no value, Bolzano says, since if it did have, it would simultaneously be equal to 0, a and $-a$, inasmuch as it can be written in several forms, as follows:

$$\begin{aligned} a - a + a - a + \cdots \\ &= (a - a) + (a - a) + \cdots \\ &= a + (-a + a) + (-a + a) + \cdots \\ &= (-a + a) + (-a + a) + \cdots \\ &= -a + (a - a) + (a - a) + \cdots. \end{aligned}$$

The author's demand, however, that a satisfactory definition for the value of a series must allow for the interchange of the order of terms and for the insertion and omission of parentheses, is of course an unreasonable one; it is not indeed satisfied even for all convergent series.

Bolzano now discusses the two series

$$S_1 = 1 + 2 + 3 + 4 + \cdots,$$
$$S_2 = 1 + 2^2 + 3^2 + 4^2 + \cdots.$$

It is pointed out, on the one hand, that $S_1 < S_2$, inasmuch as each term of S_1 (except the first) is less than the corresponding term of S_2; but on the other hand, $S_1 > S_2$, since every term of S_2 occurs somewhere in S_1 and there are terms in S_1 which do not occur in S_2. Bolzano's explanation of this paradox is that in reality $S_2 > S_1$; for one has no right to infer that the first of two infinite series has a greater value than the second simply on the ground that the first contains all the terms of the second and many other terms besides. This explanation is of course entirely incorrect. The fact is that neither of the two series can be asserted to be greater in value than the other, because as neither of them possesses a value, their values cannot be compared.

There are misstatements in the theory of functions. The author states (p. 65, footnote) that every continuous function, except possibly for isolated values of the independent variable, possesses a derivative, and (p. 68) can be expanded by Taylor's Theorem.

On the other hand Bolzano must be credited with ideas that were certainly in advance of his time. For example, he gives a proof (p. 13) of the existence of an infinite set—by proving that the number of propositions is infinite—and this anticipates Dedekind by nearly forty years. Again, he points out the possibility of setting up a one-one correspondence between the elements of an infinite set and those of a proper part of that set—an idea that was to become fundamental in the work of Cantor and Dedekind.

The book deals with paradoxes of the infinite. Some of these so-called paradoxes are, from the standpoint of present knowledge, not paradoxes at all. Such, for example, are those involving infinite series, mentioned above. Others are cleared up by an accurate application of the definitions of infinity and infinitesimal.

The book is of interest to the philosopher and to the theologian as well as to the mathematician. Bolzano was an Austrian priest, who as Professor of the Philosophy of Religion at Prague tried in his lectures to reconcile Catholic theology with the ideas of modern science. Twenty-five pages of the book are devoted to purely metaphysical considerations, which include a development of the doctrine of panpsychism and an application of this doctrine to the problem of interaction between mind and matter.

The notes by Hans Hahn constitute a helpful guide to the text. Explanations are given only to the mathematical portions of the text (pp. 1–107); the twenty-five pages of metaphysical discussion are not considered by Hahn.

L. L. Silverman

SHOTER NOTICES

Die Theorie der Gruppen von endlicher Ordnung, mit Anwendungen auf algebraische Zahlen und Gleichungen sowie auf die Kristallographie. By Andreas Speiser. Berlin, Julius Springer, 1923. viii + 194 pp.

This book is volume 5 of the series entitled *Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete*, of which the first volume appeared in 1921. In accord with the plan of the series special attention is given to applications, but the greater part of the work is devoted to the development of the fundamental theorems relating to the theory of the groups of finite order. The beginner who may find the developments too brief is referred to the second volume of Weber's *Algebra* and to Netto's *Gruppen und Substitutionentheorie* for more extensive treatments.

From the title of the series it is clear that the aim is to deal with fundamental theories rather than to present details. Differences of opinion naturally exist as regards what should be regarded as most fundamental. Some might regard the theory of the ϕ-subgroups, which does not appear in the present volume, as more fundamental than some of the theories which do appear. In fact, some readers may not agree with a statement found on page 97 to the effect that the representation of groups by means of substitutions is the most important domain of group theory. Most readers will probably agree, however, that the material of the present volume has been, on the whole, wisely selected for the purposes in view, which seem to have included an introduction to the theories due to Frobenius. From this standpoint the present volume is especially useful.

The following list of the fifteen chapter headings may serve to exhibit the nature of the material treated: The foundations, invariant subgroups and factor-groups, abelian groups, conjugate subgroups, Sylow groups and p-groups, groups of crystallography, permutation groups, automorphisms, monomial groups, representation of groups by linear homogeneous substitutions, group-characteristics, applications of the theory of group-characteristics, arithmetical researches on substitution groups, groups of a given degree, and theory of equations.

Comparatively few errors seem to have escaped the notice of the author. Among the few that might possibly trouble the beginner are the following: On page 66 it is stated, in theorem 66, that the subgroups composed separately of all the substitutions of a transitive permutation group of degree n which omit a given letter constitute a system of n conjugate subgroups of index n. This is true only when these subgroups are actually of degree $n - 1$. On page 76 it is stated that the cyclic invariant subgroup of index 2 of a dihedral group gives rise to outer isomorphisms. This is evidently not the case when the order of this dihedral group is 6, 8, or 12. The first of these groups has no outer isomorphisms at all.

G. A. Miller

Repertorium der Höheren Mathematik. E. Pascal. Second edition; second volume, *Geometry*; second section, *Geometry of Space.* Edited by H. E. Timerding. Leipzig, B. G. Teubner, 1922. xii + 628 pp.

The first section of the volume on analysis and also of the one on geometry of this second edition of Pascal's *Repertorium of Higher Mathematics* appeared in 1910. They were reviewed in this BULLETIN (volume 19, pp. 372–374). The second section of the volume on geometry, after being on the eve of publication for nearly ten years, has now appeared.

This section is devoted to geometry of space. In a subject so extensive, the first problem is a proper selection of topics for consideration. Somewhat more than half of the work is devoted to algebraic surfaces. Of the remainder, about one third is devoted to algebraic curves, a third to differential geometry and the remainder to line geometry and algebraic transformations of space. It is to be regretted that some of the more recondite portions of these subjects were not omitted in favor of the elements of synthetic, particularly of projective, geometry of space.

The treatment of the various topics is always good and occasionally excellent. In so brief and comprehensive a volume, the treatment of any subject must devote itself mainly to fundamentals and be of interest chiefly to those who are not specialists in that subject. The needs of such readers are accordingly borne constantly in mind. Occasionally, the limitations of a theorem are not adequately stated, but in some of these cases the error is due to the carelessness of the investigator to whom the theorem is due. Occasionally also, technical terms are used which are neither explained in the context nor to be found in the index.

The authors have given in clear and compact form the principal facts of the topics discussed. Mathematicians in every field will find this a useful work for reference.

C. H. SISAM

Tables Logarithmiques à treize décimales. By H. Andoyer. Paris, J. Hermann, 1922. x + 26 pp.

This pamphlet gives the logarithms of numbers from 100 to 1000 and of numbers from 100,000 to 101,000 to thirteen places besides some shorter tables of first and second differences. The tables are designed, of course, for use in work in which the usual tables to seven or eight places would not give the desired accuracy. A mechanical computer would be a great aid in the use of the tables.

In using the tables every number N is first resolved into two factors n and N' where, in general, n is taken to be the first three digits of N but in any case the three digits so that the first three digits of N' shall be 100. It remains then merely to add to the logarithm of n taken from the first table the logarithm of N' taken from the second table. The table of corrections corresponding to the first and second differences is used, of course, in obtaining the logarithm of N'. The inverse problem of finding the antilogarithm of a logarithm involves no additional principles.

C. H. FORSYTH

Fundamental Congruence Solutions. By Lt.-Col. Allan Cunningham. London, Francis Hodgson, 1923. xviii + 92 pp.

Haupt-exponents, Residue-indices, Primitive Roots, and Standard Congruences. By Lt.-Col. Allan Cunningham. Francis Hodgson, 1922. viii + 136 pp.

The first of these two books gives in 92 pages one value of the root y, of the "fundamental congruence":

$$y^{\xi} \equiv +1 \pmod{p \text{ or } p^k}$$

for values of the modulus not exceeding 10,201. ξ is the "Haupt-exponent," or the exponent to which y belongs. Owing to the immense amount of labor necessary to determine it, the smallest value of y is not always the one given. If, however, the smallest value is not greater than 13, it is given. Also when $\xi = p - 1$ (and y is therefore a primitive root), the smallest value of y is listed. There are further cases, when ξ has certain other values, where the root y is the smallest root.

One root, y, being listed, the others may be determined. Rules are given to effect this. Some account of the method of computation of the tables appears in the introduction, together with a description of the checks employed.

The tables are arranged in two lines for each modulus, thus:

p	ξ, ν
	y

where ν is the complementary factor of ξ with respect to $p - 1$, so that $\xi\nu = p - 1$. The values of the modulus are arranged consecutively so far as the primes are concerned, but for some reason, not explained, and not quite clear to the reviewer, those moduli which are powers of primes are inserted, part on the eighth page and part on the ninetieth page. These lists are not hard to find when one knows where they are, but a note telling where they are hidden would save users of the table some little annoyance and bewilderment.

In 31 pages of the second book we are given for each prime p not greater than 10,000 the factorization of $p - 1$. Then is given also the exponent to which each of the numbers 2, 3, 5, 6, 7, 10, 11, 12 belongs. A primitive root for each prime is also indicated. Then comes a similar table for the powers of 2, followed by a page giving the same information for powers of odd primes. Pages 33 to 34 give the solutions (x_0, α_0, x_0') of the two congruences

$$\left.\begin{aligned} 2^{x_0} &\equiv \pm y^{\alpha_0} \\ 2^{x_0'} y^{\alpha_0} &\equiv \pm 1 \end{aligned}\right\} \pmod{p \text{ or } p^k}$$

for $y = 3, 5, 7, 11$, and p as before. A similar table follows in which the congruences are

$$\left.\begin{aligned} 10^{x_0} &\equiv \pm y^{\alpha_0} \\ 10^{x_0'} y^{\alpha_0} &\equiv \pm 1 \end{aligned}\right\} \pmod{p \text{ or } p^k}.$$

The final table of the book, beginning at page 97, is a continuation of the first, slightly abridged to accommodate the larger numbers that appear as p runs from 10,007 to 25,409 inclusive.

The value of a piece of tabular work may appear in either or both of two ways. The table may be of use in other computations; or it may serve to put complicated results before the eye in such a simple form as to reveal hidden relations. The first of these purposes is evidently the one intended for the two books of tables published by Lt.-Col. Cunningham. They are to serve as aids in the factorization of numbers of the form

$$y^n \pm 1.$$

A detailed explanation of their use in this connection, together with illustrative examples, would do much to increase their usefulness and availability to other workers in this field. The need for an extensive table of primitive roots is readily appreciated by any one working in the theory of numbers. If one should by chance encounter a congruence such as

$$y^{356} \equiv 1 \pmod{4297}$$

he would turn to these tables with gratitude, but just how a congruence of this sort might arise in connection with other parts of the theory of numbers is a question to which at least a few words might well be given.

D. N. Lehmer

Vector Calculus. By James Byrnie Shaw. New York, D. Van Nostrand Company, 1922. iv + 314 pp.

In his preface the author says that "he has examined the various methods that go under the name of vector, and finds that for all purposes of the physicist and for most of those of the geometer, the use of quaternions is by far the simplest in theory and in practice." This indicates clearly the point of view of the book. The quaternion notation is used but tables of other equivalent notations are given.

The first chapter is a historical sketch of the various systems of vector analysis. The next six chapters are concerned with scalar and vector fields, the algebraic combinations of vectors and the differential operations. These are illustrated by a large number of quantities occurring in geometry, electricity and magnetism, mechanics, theory of elasticity, etc., each of which is defined when introduced. The following two chapters give a systematic exposition of the differential and integral calculus of vectors, with applications to geometry and such topics as Laplace's equation, Green's theorem, and spherical harmonics. The remaining three chapters treat the linear vector function with applications to deformable bodies and hydrodynamics. Extensive lists of problems are given covering almost all the topics discussed.

The book impresses one as containing an extraordinary number of topics treated in a way that (to one acquainted with those topics) is interesting and easy to follow. Students of the better class will certainly acquire a considerable knowledge of mathematics and physics by studying this book. Whether an individual teacher chooses it, however, will probably depend on whether he is willing to use the quaternion notation or translate it into the form that he does use.

H. B. Phillips

Tables of Applied Mathematics in Finance, Insurance, Statistics. By James W. Glover. Ann Arbor, Mich., George Wahr, 1923. xiii + 676 pp.

This book of tables is designed for use in connection with college textbooks on finance, insurance and statistics. The tables are quite complete and extensive for use in finance and insurance and would be very useful in many statistical investigations. The tables are divided into four parts, of which the last part, comprising almost 200 pages, is a photographic reproduction of Bruhn's table of logarithms to seven places.

Part I contains tables of compound interest functions (to eight places) and their logarithms (to seven places) for thirty-two rates of interest, those up to and including $1\frac{1}{8}$ per cent. for periods up to 200 and for higher rates up to 100. Some of the tables are published here for the first time. The tables include practically every conceivable period and rate of interest which would be involved in converting interest. There are also auxiliary tables to be used when interest is converted very frequently and even instantaneously. Tables of the sinking fund function are also included so that all of the six fundamental functions are tabulated. Logarithms of the factor $1 + i$ are also given to twenty places for certain rates of interest for computing very unusual values of functions.

Part II contains values of life insurance and disability functions, including ordinary commutation columns based on the American Experience table and at three rates of interest. Tables of functions to be used in computing terminal reserves, paid-up and extended insurance values, etc., are given for the rate 4 per cent. which are not included in Dawson's *Derived Tables.* Hunter's Makehamized table and commutation columns for two lives for work with joint lives, and Hunter's disability tables are useful divisions of this part.

Part III contains tables to be used in statistical work, some of which are excellent and very desirable. The most important are the tables of ordinates, areas and derivatives (second to eighth) of the normal curve, the logarithms of the Gamma function and squares, cubes, square and cube roots (of numbers up to 1000). One of the criticisms of the reviewer would be of the selections of some of the other tables given in this part. However, the latter tables occupy a relatively small space and it would be difficult if not impossible to select and include a set of tables for statistical work which would suit everyone.

Another criticism which applies to the whole set of tables, but particularly to this part, is of the lack of explanation of the functions tabulated. A short reference in the preface to a particular use to be made of certain ones of the tables is, with about one exception, about all the explanation offered. A short explanation and an application of each table given in this part (similar to those given in the preface of Pearson's *Tables for Statisticians*, etc.), given preferably on the page just preceding the table, would add much to the usefulness of the book. As a single example, the use of the page of logarithms of Bernoulli's numbers from $n = 1$ to $n = 200$ will probably be very much affected by the lack of an explanation, and some who have used Bernoulli's numbers before may have to make a preliminary and independent investigation to make sure of the notation used here.

The only other criticism is in regard to the omission in several cases of all signs of the negative part of a characteristic of a logarithm, especially in a table or among tables where a positive characteristic of more than 9 is not unexpected. The scheme followed in printing the tables no doubt had most to do with these omissions, but nevertheless the omissions are unfortunate.

The tables as a whole are excellent both in intrinsic value and typography, and are bound to be adopted widely as a standard in work for which they were designed. The feature which will probably commend them most is their completeness for work in finance with logarithms. Logarithms are a practical necessity in such work and a four-place table is in most cases scarcely less absurd than no table at all. Yet there is little doubt but that few of the many attempts made throughout this country to give courses in finance leave a satisfactory impression of the use of logarithms in such connections upon the mind of the student. There should be no further excuse for this kind of a situation with this book available.

The author's reputation for carefulness and reliability is enough to warrant the reliability of the tables. The author states in the preface that it is scarcely possible to compile so large a set of tables without a few errors creeping in somewhere.

C. H. Forsyth

Iamblichvs Theologovmena Arithmeticæ, edidit Victorivs de Falco. Lipsiæ in Ædibvs, B. G. Tevbneri, MCMXXII. xvii + 90 pp.

This booklet is a new edition of the Greek text of the *Theologovmena arithmeticæ* which is attributed by some critics to Iamblichus. It is edited after a careful re-examination of the various extant manuscripts and is accompanied by notes written in Latin. This publication will be of interest to students desiring to enter more intimately into the study of the relations of mathematics to philosophy and mysticism than is usually done in our histories of mathematics. It deals with the theologic aspect of numbers and their mystic relations to the various heathen deities, and with obscure cosmological speculations.

Florian Cajori

Vektoranalysis. By Siegfried Valentiner. Third edition. (Sammlung Göschen 354.) Berlin, Vereinigung Wissenschaftlicher Verleger, 1923. 132 pp.

This little manual is the slightly modified, rewritten edition of the original. The chief changes are in the use of smaller type and closer setting which has apparently reduced the amount. The only serious cut however is in the omission of the useful collection of formulas at the end of the preceding edition. Slight changes of the text occur, but none of importance.

J. B. Shaw

NOTES

At the meeting of the Southwestern Section of this Society, at the University of Missouri on December 1, Professor Henry Blumberg, of the University of Illinois will give an address on *Properties of unrestricted functions*, at the invitation of the program committee.

Three numbers of the TRANSACTIONS OF THIS SOCIETY have recently been published almost simultaneously, from three different presses. These numbers contain the following papers. Volume 23, number 4 (June, 1922): *A proof and extension of the Jordan-Brouwer separation theorem*, by J. W. Alexander; *Oscillation theorems in the complex domain*, by E. Hille; *On certain relations between the projective theory of surfaces and the projective theory of congruences*, by F. E. Wood; *Asymptotic planetoids*, by D. Buchanan; vol. 24, number 1 (July, 1922): *Associated sets of points*, by A. B. Coble; *On algebraic functions which can be expressed in terms of radicals*, by J. F. Ritt; *On the location of the roots of the jacobian of two binary forms, and of the derivative of a rational function*, by J. L. Walsh; *I-conjugate operators of an abelian group*, by G. A. Miller; volume 25, number 1 (January, 1923): *The* (1, 2) *correspondence associated with the cubic space involution of order two*, by F. R. Sharpe and V. Snyder; *Sur certaines équations aux différences finis*, by N. E. Nörlund; *Differential geometry of an m-dimensional manifold in a euclidean space of n dimensions*, by C. E. Wilder; *Expansions in terms of solutions of partial differential equations. First paper: Multiple Fourier series expansions*, by C. C. Camp; *Euler algebra*, by E. T. Bell.

The opening number of volume 45 of the AMERICAN JOURNAL OF MATHEMATICS (January, 1923) contains: *On the number of solutions in positive integers of the equation* $yz + zx + xy = n$, by L. J. Mordell; *A closed set of normal orthogonal functions*, by J. L. Walsh; *Congruences determined by a given surface*, by Claribel Kendall; *Linear partial differential equations with a continuous infinitude of variables*, by I. A. Barnett; *On the ordering of the terms of polars and transvectants of binary forms*, by L. Isserlis.

The concluding number of volume 23, series 2, of the ANNALS OF MATHEMATICS (June, 1922) contains: *On the positions of the imaginary points of inflexion and critic centers of a real cubic*, by B. M. Turner; *Frequency distributions obtained by certain transformations of normally distributed variates*, by H. L. Rietz; *The associated point of seven points in space*, by H. S. White; *Common solutions of two simultaneous Pell equations*, by A. Arwin; *On the complete independence of Hurwitz's postulates for abelian groups and fields*, by B. A. Bernstein; *On power series with positive real part in the unit circle*, by T. H. Gronwall; *Algebraic surfaces, their cycles and integrals. A correction*, by S. Lefschetz.

The following university courses in mathematics are announced for the academic year 1923–1924, in addition to those listed in the July number of this BULLETIN:

University of Wisconsin (First Semester).—By Professor E. B. Skinner: Advanced calculus; Theory of numbers.—By Professor H. W. March: Harmonic analysis.—By Professor E. B. Van Vleck: Differential equations; Theory of analytic functions; Integral equations—By Professor A. Dresden: Calculus of variations.—By Professor C. S. Slichter: Mechanics.—By Professor L. W. Dowling: Projective geometry.

Yale University.—By Professor J. K. Whittemore: Differential geometry; Special topics in advanced differential geometry.—By Professor W. A. Wilson: Functions of a real variable; Special topics in the theory of aggregates.—By Professor E. J. Miles: Advanced calculus of variations. —By Professor E. W. Brown: Celestial mechanics.—By Professor J. I. Tracey: Higher algebra.—By Professor James Pierpont: Non-euclidean geometry and Einstein's theory.—By Mr. Mikesh: Teachers' course.

Drury College, on the occasion of its fiftieth anniversary in June, 1923, conferred the honorary degree of doctor of laws on Professor B. F. Finkel, of the department of mathematics, founder of the American Mathematical Monthly.

Professor G. D. Birkhoff of Harvard University received the honorary degree of doctor of science from Brown University at the June, 1923, commencement.

The following Italian professors have been transferred as indicated: Professor G. Armellini from the chair of higher mechanics at Pisa to that of astronomy at Rome; Professor A. Comessati from the chair of analytic geometry at Cagliari to that of descriptive geometry at Padua; Professor E. Laura from the chair of rational mechanics at Pavia to the same chair at Padua; Professor A. Palatini from the chair of rational mechanics at Messina to that of mathematical physics at Naples.

The following have been appointed to associate professorships: Dr. E. Bompiani, in analytic geometry, at the Milan Technical School; Dr. C. Rosati, in projective and descriptive geometry, at the University of Pisa; Dr. G. Sannia, in analytic geometry, and Dr. G. Vitali, in infinitesimal calculus, at the University of Modena.

Professor G. D. Olds of Amherst has been elected acting President of that college for the academic year 1923–1924, with the understanding that he will be made president at the end of the year. Professor Olds has been professor of mathematics at Amherst since 1891, was dean from 1908 to 1920, and acting president 1920–1921.

Associate Professor R. C. Archibald of Brown University has been promoted to a full professorship of mathematics.

Professor P. J. Daniell, of Rice Institute, has been appointed to the Town Trust chair of mathematics at the University of Sheffield.

Captain D. M. Garrison, of the corps of Professors of Mathematics, U. S. Navy, is retiring from the Navy, and has accepted the professorship of mathematics at St. John's College, Annapolis, Md. Captain Garrison has been a member of the department of mathematics at the U. S. Naval Academy for twenty years, and for the past five years, during the difficult period of readjustment following the World War, has been head of the department. He will be succeeded as head of the department by Commander A. J. Chantry, Jr., of the corps of Naval Constructors.

Mr. P. Y. Yoder, of the University of Kansas, has been appointed professor of mathematics and physics at Blue Ridge College, Maryland.

Professor J. V. DePorte, of the New York State College for Teachers, has been granted leave of absence for the academic year 1923–24, which he will spend at Johns Hopkins University.

Dr. C. H. Yeaton and Dr. F. E. Carr have been appointed assistant professors of mathematics at Oberlin College.

Professor C. N. Mills, of Heidelberg University, Tiffin, Ohio, has accepted a professorship of mathematics at the South Dakota State Normal School.

Mr. Norman Anning, instructor in mathematics at the University of Michigan, has been promoted to an assistant professorship.

Dr. Frederick Wood, instructor in mathematics at the University of Wisconsin, has accepted a professorship of mathematics in the State Normal School, Indiana, Pennsylvania.

At the University of Iowa, Assistant Professor W. H. Wilson is on leave of absence for the academic year 1923–1924. Mr. R. E. Kennon, instructor in mathematics, has resigned to accept a position as examiner of the Iowa State Insurance Department.

At the University of Texas, Associate Professor E. L. Dodd has been promoted to a full professorship and made chairman of the department. Associate Professor R. L. Moore has been promoted to a full professorship, Adjunct Professor H. J. Ettlinger to an associate professorship, and Dr. P. M. Batchelder and Miss Mary Decherd to adjunct professorships.

Assistant Professor E. W. Pehrson, of the University of Utah, has been promoted to an associate professorship of mathematics.

The following appointments to instructorships in mathematics are announced: Mr. H. E. Arnold and Mr. G. W. Bain, at Wesleyan University; Mr. J. P. Ballantine, of the University of Chicago, at Columbia University; Dr. I. A. Barnett, of the University of Saskatchewan, at the University of Cincinnati; Mr. H. W. Chandler, of the University of Minnesota, at the University of Florida; Mr. T. H. Milne, of the University of Alberta, at the University of Buffalo; Dr. Jesse Osborn, of Cornell University, at the University of Iowa. Dr. I. Maizlish, of the University of Minnesota, has been appointed instructor in physics at Lehigh University.

Dr. C. A. Garabedian, of Harvard University, has received an appointment as Parker Traveling Fellow for the academic year 1923–1924.

Professor W. Killing died February 11, 1923, at the age of seventy-two years.

Professor S. D. Killam, of the University of Alberta, was drowned in Lake Annis, Nova Scotia, July 22, 1923.

Dr. T. M. Blakslee, of Des Moines College, died January 30, 1923.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

BERZOLARI (L.). Geometria analitica. II: Curve e superficie del secondo ordine. 2a edizione riveduta ed ampliata. (Manuali Hoepli.) Milano, Hoepli, 1922. 10 + 474 pp.

BIEBERBACH (L.). Theorie der Differentialgleichungen. (Die Grundlehren der mathematischen Wissenschaften, Band 6.) Berlin, Springer, 1923. 8 + 320 pp.

BREWSTER (G. W.). Common sense of the calculus. Oxford, Clarendon Press, 1923. 62 pp.

BROAD (C. D.). Scientific thought. London, Kegan Paul, and New York, Harcourt, Brace and Company, 1923. 555 pp.

BROWNLEE (J.). Log $\Gamma(x)$ from $x = 1$ to 50.9 by intervals of .01. (Tracts for Computers, No. 9.) London, Cambridge University Press, 1923. 23 pp.

DE COMBEROUSSE (C.). Cours de mathématiques. Tome 4: Algèbre supérieure. 2e partie, entièrement refondue par R. de Montessus de Ballore. Paris, Gauthier-Villars, 1923. 16 + 1033 pp.

DARBOUX (G.). Leçons sur la théorie des surfaces. Partie 3: Lignes géodésiques. Paramètres différentiels. Déformation des surfaces. Nouveau tirage. Paris, Gauthier-Villars, 1923.

DICKSON (L. E.). Algebras and their arithmetics. Chicago, University of Chicago Press, 1923. 241 pp. $2.35

DICKSON (L. E.), MITCHELL (H. H.), VANDIVER (H. S.), and WAHLIN (G. E.). Algebraic numbers. (Bulletin of the National Research Council.) Washington, National Academy of Sciences, 1923. 96 pp. $1.50

ENCYKLOPÄDIE der mathematischen Wissenschaften. Band III 3, Heft 6: R. Weitzenböck, Neuere Arbeiten der algebraischen Invariantentheorie. Differentialinvarianten. Leipzig, Teubner, 1922.

EUDOXE. Géométrie pure et géométrie synthétique. Paris, A. Blanchard, 1923. 8vo. 80 pp.

FUETER (R.). Synthetische Zahlentheorie. Neue Ausgabe. Berlin, Vereinigung wissenschaftlicher Verleger, 1921. 8 + 271 pp.

GAUSS (C. F.). Werke. Herausgegeben von der Gesellschaft der Wissenschaften zu Göttingen. Band 10, Abteilung 2, 1te, 4te und 5te Abhandlung. Leipzig, Teubner, 1922–23.

HAAG (J.). Cours complet de mathématiques spéciales. Tome 4: Géométrie descriptive. Trigonométrie. Paris, Gauthier-Villars, 1923. 11 + 152 pp.

HEFFTER (L.). Lehrbuch der analytischen Geometrie. Band 2: Geometrie im Bündel und im Raum. Leipzig, Teubner, 1923. 12 + 423 pp.

Kneser (A.). Die Integralgleichungen und ihre Anwendungen in der mathematischen Physik. 2te, umgearbeitete Auflage. Braunschweig, Vieweg, 1922. 8 + 292 pp.

Krull (W.). Ueber Begleitmatrizen und Elementarteilertheorie. (Diss.) Freiburg i. Br., 1922.

de Launay (L.). Les grands hommes de France: Descartes. Paris, Payot, 1923.

Loney (S. L.). The elements of coordinate geometry. Part 2: Trilinear coordinates. London, Macmillan, 1923. 8 + 228 pp.

Mitchell (H. H.). See Dickson (L. E.).

de Montessus de Ballore (R.). See de Comberousse (C.).

Muir (T.). The theory of determinants in the historical order of development. Volume 4: The period 1880 to 1900. London, Macmillan, 1923. 31 + 508 pp.

Nagaoka (H.) and Sakurai (S.). Tables of theta functions, elliptic integrals K and E, and associated coefficients in the numerical calculation of elliptic functions. (Scientific Papers of the Institute of physical and chemical Research, Volume II.) Tokyo, 1922.

National Committee on Mathematical Requirements. The reorganization of mathematics in secondary education, a report by the National Committee on Mathematical Requirements under the auspices of the Mathematical Association of America. Hanover, N. H., Mathematical Association of America, 1923. 11 + 652 pp.

Nielsen (N.). Traité élémentaire des nombres de Bernoulli. Paris, Gauthier-Villars, 1923. 398 pp.

Pasch (M.). Die Begriffswelt des Mathematikers in der Vorhalle der Geometrie. Leipzig, Meiner, 1922.

Riemann (B.). Ueber die Hypothesen, welche der Geometrie zu Grunde liegen. Neu herausgegeben und erläutert von H. Weyl. 3te Auflage. Berlin, Springer, 1923. 6 + 48 pp.

Sakurai (S.). See Nagaoka (H.).

Study (E.). Einleitung in die Theorie der Invarianten linearer Transformationen auf Grund der Vektorenrechnung. 1ter Teil. Braunschweig, Vieweg, 1923. 2 + 268 pp.

Tropfke (J.). Geschichte der Mathematik. 2te, verbesserte und sehr vermehrte Auflage. Band 4: Ebene Geometrie. Berlin, Vereinigung wissenschaftlicher Verleger, 1923. 238 pp.

de la Vallée Poussin (C.). Cours d'analyse infinitésimale. 4e édition. Tome 2. Paris, Gauthier-Villars, 1922. 15 + 478 pp.

Vandiver (H. S.). See Dickson (L. E.).

de Vries (J. F.). Analytische behandeling van de rationale kromme van den vierden graad in een vierdimensionale ruimte. s' Gravenhage, Nijhoff, 1922. 11 + 159 pp.

Wahlin (G. E.). See Dickson (L. E.).

Weitzenböck (R.). See Encyklopädie.

Weyl (H.). See Riemann (B.).

Witting (A.). Funktionen, Schaubilder und Funktionstafeln. Eine elementare Einführung in die graphische Darstellung und in die Interpolation. Leipzig, Teubner, 1922.

PART II. APPLIED MATHEMATICS

ANNUAIRE pour l'an 1923 publié par le Bureau des Longitudes. Paris, Gauthier-Villars, 1923. 872 pp.

ARISTOTLE. The works of Aristotle translated into English. De Cælo, by J. L. Stocks. Oxford, University Press, 1922.

BAUER (E.). La théorie de la relativité. Paris, Librairie de l'Enseignement technique, 1922. 12mo. 4 + 128 pp.

BELL (L.). The telescope. London, McGraw-Hill, 1922. 9 + 287 pp.

BIGELOW (F. H.). Atmospheric radiation, electricity and magnetism. Vienna, 1922. 89 pp.

BJORCK (S.). Le tourbillon cartésien. Ses conséquences. Stockholm, 1922. 8vo. 15 pp.

BOUASSE (H.). La question préalable contre la théorie d'Einstein. Paris, Blanchard, 1923. 28 pp.

TYCHONIS BRAHE opera omnia. Tomi quinti fasciculus posterior. Edidit I. L. E. DREYER. Hauniæ, Libraria Gyldendaliana, 1923. 127 pp.

BROSE (H. L.). See SOMMERFELD (A.).

CAMPBELL (N. R.). Modern electrical theory. Supplementary chapters. Chapter 16: Relativity. Cambridge, University Press, 1923. 8 + 116 pp.

CASAZZA (G.). I principî della meccanica alla luce della critica. Roma, Albrighi, Segati e Cia., 1921. 174 pp.

CHRISTESCO (S.). La lumière relative et l'expérience de Michelson. Paris, Alcan, 1923. 8vo. 48 pp.

DUBROCA (M.). L'erreur de M. Einstein. L'inacceptable théorie. L'éther et le principe de la relativité. Paris, Gauthier-Villars, 1922. 8vo. 48 pp.

EDDINGTON (A. S.). The mathematical theory of relativity. Cambridge, University Press, 1923. 10 + 248 pp.

ENNIS (W. D.). Thermodynamics, abridged. 2d edition, corrected. New York, 1922. 244 pp. $4.00

FLEMING (J. A.). Waves and ripples on water, air and æther. 4th issue, revised. London, Sheldon Press, and New York, Macmillan, 1923. 12 + 299 pp.

FOURNIER (G.). La relativité vraie et la gravitation universelle. Paris, Gauthier-Villars, 1923. 8 + 130 pp.

FRITSCHE (J.). Die Berechnung des symmetrischen Stockwerkrahmens mit geneigten und lotrechten Ständern mit Hilfe von Differenzengleichungen. Berlin, Springer, 1923. 6 + 90 pp.

GLAZEBROOK (R.). A dictionary of applied physics. Volume 3: Meteorology, metrology and measuring apparatus. Volume 4: Light, sound, radiology. London, Macmillan, 1923. 7 + 839 + 8 + 914 pp.

GÖTZE (R.). See PASCHEN (F.).

GRÄTZ (L.). Die Atomtheorie in ihrer neuesten Entwickelung. Sechs Vorträge. Stuttgart, Engelhorn, 1921. 8 + 93 pp.

HAMMICK (D. L.). See PERRIN (J.).

HUND (A.). Hochfrequenzmesstechnik. Ihre wissenschaftlichen und praktischen Grundlagen. Berlin, Springer, 1922.

JÄGER (G.). Theoretische Physik. II–III: 5te Auflage. Berlin, Vereinigung wissenschaftlicher Verleger, 1921–22.

KÖNIG (A.). Die Fernrohre und Entfernungsmesser. Berlin, Springer, 1923. 8vo. 7 + 207 pp.

LAUDIEN (K.). Leitfaden der Mechanik für Maschinenbauer. Berlin, Springer, 1921.

LEUSCHNER (A. O.). Celestial mechanics. A survey of the status of the determination of the general perturbations of the minor planets. (Bulletin of the National Research Council.) Washington, National Academy of Sciences, 1922. 73 pp.

MÜLLER (E.). Lehrbuch der darstellenden Geometrie für technische Hochschulen. 3te Auflage. Band 2. Leipzig, Teubner, 1923. 10 + 362 pp.

PACORET (E.). Les forces hydrauliques et les usines hydroélectriques. Paris, Delagrave, 1923. 452 pp.

PARISELLE (H.). Les instruments d'optique. Paris, Colin, 1923. 6 + 218 pp.

PASCHEN (F.) und GÖTZE (R.). Seriengesetze der Linienspektren. Berlin, Springer, 1922. 4 + 154 pp.

PERRIN (J.). Atoms. Authorized translation by D. L. Hammick. 2d English edition. London, Constable, 1923. 15 + 231 pp.

PUTNAM (T. M.). Mathematical theory of finance. New York, Wiley, 1923. 10 + 117 pp.

RICCI (U.). Il fallimento della politica annonaria. Firenze, Società editrice La Voce, 1921. 8 + 494 pp.

ROY (L.). L'electrodynamique des milieux isotropes en repos d'après Helmholtz et Duhem. (Collection Scientia.) Paris, Gauthier-Villars, 1923. 94 pp.

SCHÜTZE (H.). Die mathematischen Grundlagen der Lebensversicherung. Leipzig, Teubner, 1922. 4 + 48 pp.

SHAW (N.). The air and its ways. The Rede Lecture (1921) in the University of Cambridge, with other contributions to meteorology for schools and colleges. Cambridge, University Press, 1923. 20 + 237 pp.

SOMMERFELD (A.). Atomic structure and spectral lines. Translated from the third German edition by H. L. Brose. London, Methuen, 1923. 13 + 626 pp.

SPURGEON (E. F.). Life contingencies. London, C. and E. Layton, 1922. 26 + 477 pp.

STOCKS (J. L.). See ARISTOTLE.

VANCE (R.). Business and investment forecasting. New York, Brookmire Economic Service, 1922. 132 pp.

VIEILLARD (P.). Longueurs d'onde et propagation. Paris, Gauthier-Villars, 1921. 12 + 416 pp.

WILLSON (R. W.). Laboratory astronomy. London, Oxford University Press, 1923. 8vo. 189 pp.

THE JOSIAH WILLARD GIBBS LECTURESHIP

The Council of the Society has sanctioned the establishment of an honorary lectureship to be known as the Josiah Willard Gibbs Lectureship. The lectures are to be of a popular nature on topics in mathematics or its applications, and are to be given by invitation under the auspices of the Society. They will be held annually or at such intervals as the Council may direct. It is expected that the first lecture will be delivered in New York City during the winter of 1923–24, and a committee has been authorized to inaugurate the lectures by choosing the first speaker and making the necessary arrangements.

R. G. D. Richardson,
Secretary.

AN ELECTROMAGNETIC THEORY OF LIGHT-DARTS

BY H. BATEMAN

1. *Introduction.* Since Einstein's theory of light-quanta * is being carefully tested in many ways by ingenious experiments and is gaining favor, it is thought by some physicists that in the future Maxwell's equations will no longer be regarded as the best starting point for the elucidation of physical phenomena but will be derived from the equations of quantum-theory by some limiting process.

On the other hand, it may be argued that physicists have been led to this point of view because they refuse to believe what is clearly indicated by some of the most interesting solutions of Maxwell's equations. These solutions show that light-darts * of an electrical nature may have a real existence and that it is not impossible for electricity to travel with the velocity of light.

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* A. Einstein, Annalen der Physik, vol. 17 (1905), p. 132; Physikalische Zeitschrift, vol. 10 (1909), p. 185; vol. 18 (1917), p. 121.

The chief difficulty in developing an electromagnetic theory of light-darts is that of the location of the energy of a light-dart. It is not certain that we can speak of a definite location of energy. Thus in electrostatics the energy can be expressed either as an integral of type † $\frac{1}{2}\int V de$ taken over the electric charges, or as an integral over the whole of space in which the density of energy is proportional to the square of the force.

The latter expression is usually preferred in forming a physical picture because the density of energy cannot be negative, while there is just a possibility that V and de might have opposite signs. If, however, this possibility is excluded ‡ we can regard the energy either as spread throughout the whole of space or as localized in the charges. The structure of the universe may, indeed, be such that it may be pictured in more than one way, and it is quite possible that electromagnetic theory and the theory of quanta are two different methods of calculation which will prove eventually to be equivalent, though different distributions of energy are considered in the two cases.

2. *Field Travelling in One Direction.* To make this point of view clear, let us consider an electromagnetic field specified by field-vectors E and H with components

$$(1)\begin{cases} E_x = 0, & E_y = \dfrac{\partial \Omega}{\partial y} f\left(t - \dfrac{x}{c}\right), \quad E_z = \dfrac{\partial \Omega}{\partial z} f\left(t - \dfrac{x}{c}\right), \\ H_x = 0, & H_y = -\dfrac{\partial \Omega}{\partial z} f\left(t - \dfrac{x}{c}\right), \quad H_z = \dfrac{\partial \Omega}{\partial y} f\left(t - \dfrac{x}{c}\right), \end{cases}$$

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* The term light-dart is recommended by Silberstein, PHILOSOPHICAL MAGAZINE, vol. 44 (1922), pp. 257, 956.

† In this integral V represents the electrostatic potential and de an element of charge.

‡ V and de will always have the same sign if charges of opposite signs are kept far enough apart and a large number of charges of one sign are never close to one of the opposite sign.

where Ω is a function of y and z representing the logarithmic potential of a distribution of line charges bounded by a cylindrical surface $\psi(y, z) = 0$. We may thus write

$$(2) \qquad \begin{cases} \dfrac{\partial^2\Omega}{\partial y^2} + \dfrac{\partial^2\Omega}{\partial z^2} = 0, & \psi(y, z) > 0 \\ \qquad\qquad\quad = \sigma(y, z), & \psi(y, z) < 0. \end{cases}$$

It is easily seen that the fundamental equations of the theory of electrons are satisfied with the above expressions for the components of E and H if the electricity in the field has a volume density ρ given by

$$(3) \qquad \begin{cases} \rho = 0, & \text{if} \quad \psi(y, z) > 0, \\ \rho = \sigma(y, z) f\left(t - \dfrac{x}{c}\right), & \text{if} \quad \psi(y, z) < 0, \end{cases}$$

and moves with the velocity of light in the direction of the axis of x. The energy in the field is

$$(4) \qquad \begin{aligned} &\tfrac{1}{2}\int (E^2 + H^2)dxdydz \\ &\qquad = \int\left[\left(\frac{\partial\Omega}{\partial y}\right)^2 + \left(\frac{\partial\Omega}{\partial z}\right)^2\right]\left[f\left(t - \frac{x}{c}\right)\right]^2 dxdydz \\ &\qquad = -\int \Omega\sigma\left[f\left(t - \frac{x}{c}\right)\right]^2 dxdydz, \end{aligned}$$

where the latter integral is taken over the charges in the field, the integrals

$$\int\left[\left(\frac{\partial\Omega}{\partial y}\right)^2 + \left(\frac{\partial\Omega}{\partial z}\right)^2\right] dydz \qquad \text{and} \qquad -\int \Omega\sigma dydz$$

being equal on account of the above-mentioned property of an electrostatic field if we suppose that the integral $\int \sigma dydz$ is zero.

Regarding the field (E, H) as representing waves of light and the electricity in the field as an accompanying light-dart composed of electric doublets, we can say that the energy can be regarded either as located in the field or as associated entirely with the light-dart. It should be noticed that the energy may be finite, although electricity moves with the velocity of light. Moreover, when $\sigma(y, z)$ is positive Ω will be negative, if charges of opposite signs are not too close together, and each element of the integral (4) will be positive.*

3. *First Example of a Field Radiating in All Directions*. The field just considered is a limiting case of several types of radiant field of a more general nature. As a first example, let us take the field

$$(5)\quad \left\{\begin{array}{l} E_x = -\dfrac{XZ}{RP^2}f'\left(t-\dfrac{R}{c}\right), \qquad E_y = -\dfrac{YZ}{RP^2}f'\left(t-\dfrac{R}{c}\right), \\ \qquad\qquad E_z = \dfrac{1}{R}f'\left(t-\dfrac{R}{c}\right), \\ H_x = \dfrac{Y}{P^2}f'\left(t-\dfrac{R}{c}\right), \qquad H_y = -\dfrac{X}{P^2}f'\left(t-\dfrac{R}{c}\right), \\ \qquad\qquad H_z = 0, \end{array}\right.$$

where

$$X = x - \xi, \qquad Y = y - \eta, \qquad Z = z - \zeta,$$
$$P^2 = X^2 + Y^2, \qquad R^2 = X^2 + Y^2 + Z^2,$$

and where $f(\tau)$ is an arbitrary function of τ, $f'(\tau)$ being used to denote its derivative with respect to τ.

The point Q with coordinates (ξ, η, ζ) is a primary singularity from which secondary singularities, carrying electric charges, are radiated with the velocity of light in directions parallel to the axis of z.

Take a series of such points Q on a sphere of radius a with center at the origin, and let us find the mean values of E and H for points Q on this sphere.

Using a well known theorem relating to the mean value of a wave-function over the surface of a sphere † we find that at points outside the cylinder $x^2 + y^2 = a^2$, the mean values of the field-vectors have components

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* It may be remarked also that the whole momentum may be regarded as associated with the light-dart. It is in the direction of the axis of x and its magnitude may be derived from that of the energy by dividing by c. The amount of energy associated with a light-dart will depend, of course, on the properties of the atom from which the energy was radiated and there is no reason why it should not be equal to $h\nu$, where ν is the frequency associated with the function f and h is Planck's constant.

† See Rayleigh's *Sound*, vol. II, Appendix.

$$(6)\quad \begin{cases} \overline{E}_x = -\dfrac{xz}{rp^2}F\left(t - \dfrac{r}{c}\right), \qquad \overline{E}_y = -\dfrac{yz}{rp^2}F\left(t - \dfrac{r}{c}\right), \\ \overline{E}_z = \dfrac{1}{r}F\left(t - \dfrac{r}{c}\right), \\ \overline{H}_x = \dfrac{y}{p^2}F\left(t - \dfrac{r}{c}\right), \qquad \overline{H}_y = -\dfrac{x}{p^2}F\left(t - \dfrac{r}{c}\right), \\ \overline{H}_z = 0, \end{cases}$$

$$p^2 = x^2 + y^2, \qquad r^2 = x^2 + y^2 + z^2,$$

where

$$F\left(t - \frac{r}{c}\right) = \frac{c}{2a}\left[f\left(t - \frac{r}{c} + \frac{a}{c}\right) - f\left(t - \frac{r}{c} - \frac{a}{c}\right)\right].$$

The mean values of E and H inside the cylinder are more difficult to find but we shall not need them.

The mean values for concentric spheres within the first one can be found in the same way, and we may infer that for a radially symmetric distribution of primary singularities inside the first sphere, the field outside the cylinder $x^2 + y^2 = a^2$ is given by components of the above type but with a different function F.

The density of energy in the field is

$$\frac{1}{p^2}\left[F\left(t - \frac{r}{c}\right)\right]^2,$$

and so the field energy outside the cylinder is given by the expression

$$2\pi\int_0^{\pi}\frac{d\theta}{\sin\theta}\int_{a\csc\theta}^{\infty}\left[F\left(t - \frac{r}{c}\right)\right]^2 dr.$$

Let

$$\left[F\left(t - \frac{r}{c}\right)\right]^2 = -c\frac{d}{dr}\Phi\left(t - \frac{r}{c}\right),$$

where $\Phi(-\infty) = 0$; then the integral, which we shall call I, takes the form

$$I = 2\pi c \int_0^\pi \frac{d\theta}{\sin\theta}\Phi\left(t - \frac{a}{c\sin\theta}\right).$$

Integrating by parts, we find that *

$$(7)\quad I = -2\pi a \int_0^\pi \frac{\cos\theta}{\sin^2\theta}\log\left(\tan\frac{\theta}{2}\right)\left[F\left(t - \frac{a}{c\sin\theta}\right)\right]^2 d\theta.$$

By putting $z = a$ ctn θ, this integral may be transformed into an integral taken over the boundary of the cylinder, and finally into an integral over the region inside the cylinder. Hence it follows that the whole energy can be expressed as an integral taken over the region occupied by the electric charges, and so can be associated with the light-dart.

4. *Second Example of a Field Radiating in All Directions.* The field (5) is a particular case of a more general type of field in which the primary singularity (ξ, η, ζ) moves in an arbitrary manner with velocity less than c and the direction of projection of the electric charges forming the light-darts varies with the time.

By superposing different fields of this general type other types of field may be derived by a process analogous to differentiation. The following type of field is of special interest:

Let τ be defined in terms of x, y, z and t by the equation

$$[x - \xi(\tau)]^2 + [y - \eta(\tau)]^2 + [z - \zeta(\tau)]^2 = c^2(t-\tau)^2, \qquad \tau \leqq t,$$

and let

$$(8)\quad \begin{cases} H_x = \dfrac{\partial(\sigma,\tau)}{\partial(y,z)}, & H_y = \dfrac{\partial(\sigma,\tau)}{\partial(z,x)}, & H_z = \dfrac{\partial(\sigma,\tau)}{\partial(x,y)}, \\ E_x = \dfrac{1}{c}\dfrac{\partial(\sigma,\tau)}{\partial(x,t)}, & E_y = \dfrac{1}{c}\dfrac{\partial(\sigma,\tau)}{\partial(y,t)}, & E_z = \dfrac{1}{c}\dfrac{\partial(\sigma,\tau)}{\partial(z,t)}, \end{cases}$$

$$\sigma = \frac{U'}{U}, \qquad U = lX + mY + nZ - cpT,$$

$$U' = l'X + m'Y + n'Z - cp'T,$$

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* The flow of momentum may be represented by a similar integral with $(1/c)\cos^2\theta$ instead of $\cos\theta$, for we need consider only the z-component of momentum, the x- and y-components of the total flow of momentum being zero by symmetry. It should be noticed that each element of I is positive.

where

$$X = x - \xi, \qquad Y = y - \eta, \qquad Z = z - \zeta, \qquad T = t - \tau,$$

and where l, m, n and p are functions of τ connected by the relation

$$l^2 + m^2 + n^2 = p^2.$$

The symbols l', m', n', p' are used to denote the derivatives of l, m, n and p with respect to τ.

The field thus specified has a primary singularity which moves according to the law

$$x = \xi(\tau), \qquad y = \eta(\tau), \qquad z = \zeta(\tau).$$

From this primary singularity a secondary singularity is projected in the direction specified by the direction cosines $(l/p, m/p, n/p)$, and a field radiates from the different positions of the primary singularity. The portion of the field associated with any range of values of τ vanishes completely when l, m, n and p are constant for this range of values of τ. If the direction of the singular ray oscillates as time varies, the light-dart will assume a wavy form and will possess a certain type of polarization.

It is easily seen that E and H are perpendicular and equal in magnitude while the density of energy is found to be

$$\frac{c^2T^2}{U^2}\frac{l'^2 + m'^2 + n'^2 - p'^2}{(\xi' X + \eta' Y + \zeta' Z - c^2 T)^2} = \frac{l'^2 + m'^2 + n'^2 - p'^2}{c^2(lx + my + nz - pr)^2},$$

when

$$\xi = \eta = \zeta = 0.$$

When we consider a number of such fields whose primary singularities are spread, with radial symmetry,* over a sphere of radius a, so that the singular rays at each instant all point in the same direction, we obtain a total field of the same general nature as (8) for points outside a certain tubular* surface S. The function p may, however, have a different value. The energy outside S is now expressed by the integral

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* The primary singularities are now taken to be stationary, so that ξ', η', and ζ' are zero.

$$I = \frac{1}{c^2}\int_0^\pi \sin\theta d\theta \int_{r(\theta)}^\infty dr \int_0^{2\pi} d\phi \frac{l'^2 + m'^2 + n'^2 - p'^2}{(1 - \cos\omega)^2},$$

where ω denotes the angle between the radius specified by the polar coordinates (θ, ϕ) and the singular ray corresponding to the value $t - r/c$ of τ.

To obtain an integrable case let us write

$$l = p \sin\alpha \cos\beta, \qquad m = p \sin\alpha \sin\beta, \qquad n = p \cos\alpha,$$

where α is constant and β is a function of $t - r/c$. We may now write

$$\cos\omega = \cos\alpha \cos\theta + \sin\alpha \sin\theta \cos(\phi - \beta).$$

Integrating with respect to ϕ, we obtain

$$I = \frac{2\pi}{c^2}\int_0^\pi \sin\theta d\theta \int_{r(\theta)}^\infty dr \frac{p^2\beta'^2 \sin^2\alpha(1 - \cos\alpha\cos\theta)}{(\cos\alpha - \cos\theta)^3}.$$

We may now integrate with respect to r by writing

$$p^2\beta'^2 = \psi'\left(t - \frac{r}{c}\right),$$

where

$$\psi(-\infty) = 0.$$

We thus obtain

$$I = \frac{2\pi \sin^2\alpha}{c}\int_0^\pi \sin\theta d\theta \frac{1 - \cos\alpha\cos\theta}{(\cos\alpha - \cos\theta)^3}\psi\left[t - \frac{r(\theta)}{c}\right],$$

where $r(\theta)$ is the value of r on the boundary of the tubular surface, which we assume to be a surface of revolution around the axis of z. Integrating by parts with respect to θ, we obtain an integral involving the original function $p^2\beta'^2$. This integral

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* The cross section of the tube will increase with the distance from the origin. It will be convenient to regard the tube as a surface of revolution; it will then not generally be completely filled with electric charges, but this does not matter.

$$\frac{2\pi \sin^2 \alpha}{c} \int_0^\pi p^2\beta'^2 d\theta \left[\frac{1}{2}\frac{\sin^2 \alpha}{(\cos \alpha - \cos \theta)^2} + \frac{\cos \alpha}{\cos \alpha - \cos \theta}\right] \sin \theta$$

consists only of positive terms, since $\alpha < \theta$, and it may be regarded as an integral over the boundary of the tubular surface. Thus the energy outside the tubular surface can be expressed as an integral over the boundary of the tubular surface. This means that the whole of the field energy can in one sense be regarded as located within a narrow region.

5. *Conclusion.* There may, of course, be a difference of opinion as to the physical interpretation of these results. A point of view that seems fairly reasonable is that the energy calculated is really in the field when the light-dart produces a field, but is in the light-dart when the light-dart fails to produce a field owing to the absence of ether-particles in the immediate neighborhood of the primary singularity when the light-dart is being emitted from it. The mathematical expressions for the field-vectors indicate in fact that the field is produced at the primary singularity, but they are based on the assumption that the ether is a continuous medium, whereas in reality the ether is probably discontinuous and made up of entities analogous to light-darts. Quite possibly it is only those light-darts which fail to produce fields and escape collisions during emission that have sufficient available energy to produce the photoelectric effect.

This investigation can be regarded as belonging to that class of researches in which the idea of a definite location of energy is discussed by considering the possibility of an additive distribution of stress, energy, and momentum, in which the total energy and momentum are zero and a radiation of positive energy in some directions is balanced by a radiation of negative energy in other directions, either at the same time or at a slightly different time. One type of tensor specifying such a distribution is easily written down. (See MESSENGER OF MATHEMATICS, vol. 52 (1922), p. 125.)

CALIFORNIA INSTITUTE OF TECHNOLOGY

SAME LEFT CO-SET AND RIGHT CO-SET MULTIPLIERS FOR ANY GIVEN FINITE GROUP*

BY G. A. MILLER

1. *Introduction.* Let G represent any group of finite order g and let H represent any subgroup of G. It is known that a set of distinct operators $s_2, s_3, \cdots, s_\lambda$ can always be so selected that every operator of G appears once and only once in each of the following two sets of augmented co-sets of G:

$$H + Hs_2 + Hs_3 + \cdots + Hs_\lambda,$$
$$H + s_2H + s_3H + \cdots + s_\lambda H.$$

This theorem was proved by means of the theory of substitution groups in the QUARTERLY JOURNAL OF MATHEMATICS (vol. 41 (1910), p. 382). A few years later H. W. Chapman gave an abstract proof of the same theorem in the MESSENGER OF MATHEMATICS (vol. 42 (1913), p. 132). In view of the facts that this theorem relates to very fundamental properties of a group and that errors appear in the latter article, we proceed to give here another abstract proof, and to develop a few new related theorems, as well as a generalization of the theorem itself.

2. *New Proof of the Theorem.* From the theory of double co-sets † it results that all the operators of G can be represented as follows:

$$H + Hs_\alpha H, \qquad (\alpha = 2, \cdots, \gamma).$$

For a particular value of α the double co-set $Hs_\alpha H$ represents all the operators of a certain number ρ of right co-sets and the same number of left co-sets. A necessary and sufficient condition that H be invariant under G is that $\rho = 1$ for every value of α. Hence an invariant subgroup can be defined as a subgroup which gives rise to no double co-set involving more distinct operators than the subgroup does. Every right co-set contained in $Hs_\alpha H$ has the same number of operators in common with every left co-set contained therein and vice versa.

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* Presented to the Society, April 13, 1923.

† Cf. Miller, Blichfeldt and Dickson, *Finite Groups*, 1916, p. 24.

The ρ right co-sets and the ρ left co-sets involved in the double co-set $Hs_\alpha H$ for a particular value of α can be placed in a (1, 1) correspondence in $\rho!$ ways. In any one such correspondence every two corresponding co-sets have exactly h/ρ common operators. As any one of these common operators can be used for the multiplier of the right co-set and the left co-set which correspond, these multipliers can be chosen in $(h/\rho)^\rho$ ways after the correspondence between the right co-sets and the left co-sets involved in $Hs_\alpha H$ has been established. Necessary and sufficient conditions that $h/\rho = 1$ are that G can be represented as a transitive substitution group with respect to H. When G is thus represented, the subgroup composed of all the substitutions which omit a given letter involves a regular group as transitive constituent corresponding to the double co-set $Hs_\alpha H$. In particular, H can contain no subgroup besides the identity which is invariant under G, and it cannot itself be an invariant subgroup of G when $h/\rho = 1$.

A necessary and sufficient condition that the double co-set $Hs_\alpha H$ for a particular value of α contain all the operators of G which are not found in H is that the transitive substitution group of degree λ to which H gives rise be at least doubly transitive.* In this theorem it is not assumed that this substitution group is simply isomorphic with G. In fact, this is a special case of the theorem that the number of the different double co-sets $Hs_\alpha H$ is equal to the number of the transitive constituents in the subgroup composed of all the substitutions which omit a given letter of the transitive group of degree λ to which H gives rise. A proof of this theorem results directly from the fact that a right co-set is composed of all the substitutions which replace by a given other letter the omitted letter in the subgroup to which H corresponds in the transitive group of degree λ.

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* A special case of this theorem was given by W. A. Manning, *Primitive Groups*, 1921, p. 41.

The double co-set $Hs_\alpha H$ either contains all the inverses of its operators or it contains no inverse of any operator included in it, since $Hs_\alpha^{-1}H$ is composed of the inverses of the operators contained in $Hs_\alpha H$. To obtain a set of operators $s_2, s_3, \cdots, s_\lambda$ which may be used both for right co-set multipliers and also for left co-set multipliers we may therefore proceed as follows: Let s_2 be an arbitrary operator of G which is not found in H. If s_2^{-1} does not appear in Hs_2 it cannot appear in s_2H and vice versa, since the right co-sets are composed of the inverses of the operators in left co-sets and vice versa. If s_2^{-1} does not appear in Hs_2 use it for s_3. The two right co-sets Hs_2 and Hs_2^{-1} will then be composed of the inverses of the operators of the two left co-sets $s_2^{-1}H$ and s_2H. If s_2^{-1} is in Hs_2 each of the co-sets Hs_2 and s_2H is composed of the inverses of the operators found in the other. In the latter case s_3 can be any operator of G such that its inverse is not in Hs_2 and it is not found in $H + Hs_2$. We proceed now in the same way with s_3 as with s_2.

This process may be continued until all the operators of G have been exhausted. If this were not the case we would reach a point when each of the two sets of augmented co-sets

$$H + Hs_2 + \cdots + Hs_\rho,$$
$$H + s_2H + \cdots + s_\rho H$$

would be composed of the inverses of the operators found in the other and each of the operators of G which does not appear in the former would have its inverse therein. Hence all of these operators would appear in

$$H + s_2H + \cdots + s_\rho H.$$

Since each of the operators of G appears in one or the other of the two given augmented ρ co-sets it results that if there are any additional right co-sets they can have no operator in common with any one of the additional left co-sets.

An additional right co-set could therefore not appear in a double co-set with respect to H which would involve an additional left co-set. If one of any two of the given corresponding $\rho - 1$ co-sets appears in such a double co-set the other must also appear therein. Hence it results that if any double co-set with respect to H should involve an additional right co-set it would also involve an additional left co-set. As this is impossible it follows that no additional co-set exists and hence $\rho = \lambda$.

3. *Related Theorems.* We have incidentally proved the following theorem: *If the multipliers of the right co-sets, or of the left co-sets, with respect to the same subgroup are so chosen that the inverse of a multiplier is used as the multiplier of the following co-set whenever a co-set does not involve this inverse, and that the other multipliers satisfy the condition that neither they nor their inverses appear in any preceding co-set, then it cannot happen that there are remaining operators if all the inverses of such operators appear in the preceding co-sets.*

Each of the double co-sets with respect to H is evidently invariant under H. In fact, the ρ right co-sets which appear in $Hs_\alpha H$ constitute a complete set of conjugates under H, and this is also true of the ρ left co-sets which appear therein. To prove this fundamental theorem it is only necessary to note the conjugates of Hs_α under H. The number of these may be found by first finding the subgroup of H composed of all its operators which transform s_α into some operator of Hs_α. If $t_1, t_2, \cdots, t_k$ are the operators of this subgroup and if

$$s_\alpha, h_2 s_\alpha, \cdots, h_k s_\alpha$$

are the distinct operators into which s_α is transformed under this subgroup, then s_α is also transformed into

$$s_\alpha, s_\alpha h_2, \cdots, s_\alpha h_k$$

under H. That is, s_α is transformed into as many distinct operators in $s_\alpha H$ as in Hs_α by the operators of H, and vice versa.

The ρ right co-sets $Hs_2, \cdots, Hs_{\rho+1}$ which form a complete set of conjugates under H evidently include $s_2 H$ and hence they include the ρ conjugates of s_2H under H. Hence we have the following theorem: *The double co-set $Hs_\alpha H$ is composed of the right co-sets which form a complete set of conjugates under H of the right co-set Hs_α. It coincides also with the complete set of conjugates under H of the left co-set $s_\alpha H$.*

4. *Generalized Theorem.* A double co-set of G with respect to two distinct subgroups does not have all the properties noted above of the double co-set with respect to a single subgroup. In particular, the former may involve the inverses of some of its operators without involving the inverses of all of them, as can readily be verified. On the other hand, every right co-set involved in the double co-set H_1sH_2 with respect to two distinct subgroups has the same number of operators in common with each of the left co-sets involved therein, and vice versa. This follows directly from the fact that the operators of $s^{-1}H_1s$ are equally distributed in the left co-sets with respect to H_2 of the double co-set $s^{-1}H_1s\cdot H_2$. The double co-set H_1sH_2 involves a certain number of right co-sets with respect to H_1 and a certain number of left co-sets with respect to H_2. Since the operators of each of these right co-sets are equally distributed among all of these left co-sets, and vice versa, it results that *it is always possible to select the multipliers in such a way that those of the right co-sets contained in H_1sH_2 are the same as those of the left co-sets contained therein whenever H_1 and H_2 have a common order, and that all the multipliers for the larger subgroup are included among those for the smaller when they have different orders.*

This theorem may be regarded as a double generalization of the one noted in the opening paragraph. In fact, when H_1 and H_2 have a common order but are distinct groups, the present theorem includes the former, and when H_1 and H_2 have different orders there results a further generalization. Hence the present note furnishes a generalization as well as a new proof of the theorem in question.

THE UNIVERSITY OF ILLINOIS

THE SECOND MEAN VALUE THEOREM FOR SUMMABLE FUNCTIONS*

BY M. B. PORTER

We start with the following lemma: *If $f(x)$ is summable over the interval (a, b) and hence has an indefinite integral $F(x)$ over (a, x), $a \leqq x \leqq b$, then*

$$\lim_{h\to 0}\int_a^b \frac{f(x+h)-f(x)}{h}dx = \lim_{h\to 0}\frac{F(b+h)-F(b)}{h} - \lim_{h\to 0}\frac{F(a+h)-F(a)}{h} = f(b)-f(a),$$

provided that $f(x)$ is right (or left) continuous at the ends of the interval.

Now let $\phi(x)$ denote a monotone function right (or left) continuous at a and b, and consider the identity

$$\int_a^b \frac{F(x+h)\phi(x+h)-F(x)\phi(x)}{h}dx = (\mathrm{I}) + (\mathrm{II}),$$

where

$$(\mathrm{I}) = \int_a^b \frac{F(x+h)-F(x)}{h}\phi(x+h)dx,$$

$$(\mathrm{II}) = \int_a^b F(x)\frac{\phi(x+h)-\phi(x)}{h}dx.$$

Applying a well known theorem of Lebesgue's to (I), and the first mean value theorem to (II), since we know that the expression $[\phi(x+h)-\phi(x)]/h$ is always of the same sign when $a \leqq x \leqq b$, we have

$$F(b)\phi(b) - F(a)\phi(a) + \epsilon_h' = \int_a^b f(x)\phi(x)dx + F(\xi_h)\int_a^b \frac{\phi(x+h)-\phi(x)}{h}dx + \epsilon_h'',$$

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* Presented to the Society, September 18, 1923.

where ϵ_h' and ϵ_h'' vanish with h. Applying the lemma to the last integral, we have:

$$(1)\quad F(b)\phi(b) - F(a)\phi(a) = \int_a^b f(x)\phi(x)dx + F(\xi_h)(\phi(b) - \phi(a)) + \epsilon_h,$$

where $\lim_{h\to 0} \epsilon_h = 0$.

This can be written in the form

$$\int_a^b f(x)\phi(x) = \phi(b)\int_a^{\xi_h} f(x) + \phi(a)\int_{\xi_h}^b f(x) + \epsilon_h.$$

When ϵ_h takes on the value zero, $F(\xi_h)$, which is continuous, will take on for some value ξ, lying between a and b, a value such that

$$\int_a^b f(x)\phi(x) = \phi(b)\int_a^{\xi} f(x) + \phi(a)\int_{\xi}^b f(x),$$

which is Weierstrass's form of the second mean value theorem for integrals.

If, for example, we suppose $\phi(x)$ increasing (monotonically) and replace $\phi(x)$ by A (fixed) over the interval $(a, a + k)$ and by B (fixed) over the interval $(b, b + k)$, leaving the values of $\phi(x)$ unchanged over $(a + k, b)$, we can prove in the same way that

$$\int_a^b f(x)\phi(x) = A\int_a^{\xi} f(x) + B\int_{\xi}^b f(x),$$

where

$$A \leqq \phi(x) \leqq B$$

over (a, b), by applying the same reasoning and letting the parameter k approach zero.

From this form the usual Bonnet forms of the theorem are at once deducible.

THE UNIVERSITY OF TEXAS

ANALOGIES BETWEEN THE u_n, v_n OF LUCAS AND ELLIPTIC FUNCTIONS*

BY E. T. BELL

1. *Historical Note.* The u_n, v_n of Lucas are the symmetric functions (n any real integer)

$$u_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \qquad v_n = \alpha^n + \beta^n,$$

of the roots α, β of $x^2 = px - q$, where p, q are relatively prime integers, so that

$$(u_0, u_1) = (0, 1), \qquad (v_0, v_1) = (2, p),$$

and u_n, v_n are integers satisfying the recurrence

$$x_{n+2} = px_{n+1} - qx_n.$$

The numerous remarkable properties and applications of these integers due to Lucas and others are summarized in vol. 1, chap. XVII of Dickson's *History*. From another source it is known that as early as 1878 Lucas had applied principles similar to those of his fundamental memoir † to symmetric functions of the roots of any algebraic equation and that he had obtained the connection of these, through the intermediary of elliptic and abelian functions, with the theory of numbers. This connection is still to be sought. In 1912 the writer was informed by the late C. A. Laisant, at one time a trustee of Lucas' manuscripts, that there was nowhere in them a vestige of the subject. Nevertheless Laisant recalled vividly that Lucas, about 1878, made a verbal communication to the Société Mathématique de France in which he exhibited a close isomorphism between three symmetric functions, of which one was $\alpha^n + \beta^n + \gamma^n$, of the roots α, β, γ of a cubic equation and the elliptic functions sn, cn, dn, especially as regards a species of double periodicity. All traces of this communication, also the sense in which the symmetric functions were doubly periodic, have been lost.

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* Presented to the Society, September 18, 1923.

† American Journal, vol. 1 (1878), p. 184.

Laisant also is authority for the statement that the ultimate object of Lucas in his researches on recurring series was the demonstration of Fermat's last theorem. He recalled that in 1890, shortly before Lucas' death, the latter, "in less than a quarter of an hour," reduced the proof of Fermat's theorem to the problem of showing that each of his symmetric functions had not more than two "periods." Laisant remarked that at the time he perfectly followed this reduction, which was clear and convincing. This adds another puzzle to the many already surrounding Fermat's theorem.

In his memoir Lucas insists strongly on the isomorphism between his u_n, v_n when $q = 1$ and the circular and hyperbolic functions. Later (*Théorie des Nombres*, p. 319) he obtains this isomorphism by a simpler method, deriving the initial correspondence directly from Simpson's recurrences for the sine and cosine. This particular device does not seem to be capable of extension. The significant fact, however, apparently is that the correspondence thus established is between the *multiplication* theory of the circular or hyperbolic functions and u_n, v_n. For on page 203 of his memoir he concludes a section with "the demonstration of formulas of extreme importance; for they will serve us later as the base of the theory of doubly periodic numerical functions deduced from the consideration of symmetric functions of the roots of equations of the third and fourth degrees with commensurable coefficients." These are the formulas (A), (A') below. We shall see that (A') is incorrect, but that (A), which is correct, corresponds to classical formulas in the real multiplication of elliptic functions and to recent results in the theory of complex multiplication. The problem seems to narrow down to connecting (A) in some simple manner with the symmetric function $s_n \equiv \alpha^n + \beta^n + \gamma^n$, for any solution of $x_{n+3} = px_{n+2} - qx_{n+1} + rx_n$ is of the form $a_0 s_n + a_1 s_{n+1} + a_2 s_{n+2}$, where α, β, γ are the roots of $x^3 = px^2 - qx + r$. It may be worth noting that some of the elliptic functions that occur in the analogies are precisely those which present themselves in the conversion of the square root of a polynomial of the third or fourth degree in x into a continued fraction. In this connection we may also call attention to the remarks of Lucas at the end of § X, *Théorie des Nombres*, p. 508. Possibly the analogies noted below may offer a clue to what Lucas had in mind.

2. *Analogies with Elliptic Functions.* Having proved the formula

$$(A) \qquad u_n^2 u_{m-1}u_{m+1} - u_m^2 u_{n-1}u_{n+1} = q^{n-1}u_{m-n}u_{m+n},$$

Lucas states that in the same way we have

$$(A') \qquad v_n^2 v_{m-1}v_{m+1} - v_m^2 v_{n-1}v_{n+1} = -\Delta q^{n-1}v_{m-n}v_{m+n},$$

where $\Delta = (\alpha - \beta)^2 = p^2 - 4q$, and he puts $m = n + 1$, $n + 2$ in (A), getting

$$(B) \qquad \begin{aligned} u_n^3 u_{n+2} - u_{n+1}^3 u_{n-1} &= q^{n-1}u_1u_{2n+1}, \\ u_n^2 u_{n+1}u_{n+3} - u_{n+2}^2 u_{n-1}u_{n+1} &= q^{n-1}u_2u_{2n+2}. \end{aligned}$$

He states that "the formulas (A) and (B) belong to the theory of elliptic functions, and, more especially, to the functions which Jacobi has denoted by Θ, H."

To see that (A') is incorrect we need consider only the case in which $q = 1$, so that $v_{-n} = v_n$. If m, n be interchanged in (A') the left changes sign while the right does not. Instead of (A') we should have

$$(A'') \qquad v_n^2 v_{m-1}v_{m+1} - v_m^2 v_{n-1}v_{n+1} = -\Delta^2 q^{n-1}u_{m-n}u_{m+n}.$$

To obtain the analogies with elliptic functions let

$$(1) \qquad I(a - b, a - c, a - d, \cdots) = 0$$

be an identity between the differences of $a, b, c, d, \cdots$, and define a function $f_{x,y}$ of two variables x, y by

$$(2) \qquad f_{x,y} \equiv k(f_{x,l} - f_{y,l}),$$

where $k \neq 0$ and l are arbitrary. Then (1) implies that

$$I\left(\frac{1}{k}f_{a,b}, \frac{1}{k}f_{a,c}, \frac{1}{k}f_{a,d}, \cdots\right) = 0;$$

and, in particular, from

$$(a-b)(c-d)+(b-c)(a-d)+(c-a)(b-d)=0,$$

we infer

$$f_{a,b}f_{c,d}+f_{b,c}f_{a,d}+f_{c,a}f_{b,d}=0. \tag{3}$$

If now there exist functions g_x, h_x such that

$$f_{x,\,y}=r\frac{g_{x+y}g_{x-y}}{h_xh_y}, \tag{4}$$

where r is independent of x, y, we have $g_{-x}=-g_x$, since $f_{x,\,y}=-f_{y,x}$, and (3) becomes

$$g_{a+b}g_{a-b}g_{c+d}g_{c-d}+g_{b+c}g_{b-c}g_{a+d}g_{a-d}+g_{c+a}g_{c-a}g_{b+d}g_{b-d}=0. \tag{5}$$

From its derivation (5) is a consequence of (2) and (4) alone; that is, (5) is implied by

$$\frac{g_{x-l}g_{x+l}}{h_xh_l}-\frac{g_{y-l}g_{y+l}}{h_yh_l}=\frac{g_{x+y}g_{x-y}}{kh_xh_y}, \tag{6}$$

and in particular (5) is implied by

$$g_y^2g_{x-1}g_{x+1}-g_x^2g_{y-1}g_{y+1}=g_1^2g_{x+y}g_{x-y}, \tag{7}$$

which is obtained from (6) by putting $k=l=1$, $h_x=g_x^2$. Conversely, (5) implies (7), for (5) becomes (7) when

$$(a,b,c,d)=(1,x,y,0).$$

From a given solution $g_x\equiv p_x$ of (5) and (7) we can obtain several more by devices familiar in the multiplication theory of elliptic functions. Thus, since the sum of the squares of the suffixes of the g's in (5) is $2(a^2+b^2+c^2+d^2)$ for each term,

$$g_x\equiv p_x/h_l^{x^2}$$

is also a solution of (5) and (7).

Henceforth, as in establishing the correspondence between u_n, v_n and the circular or hyperbolic functions, we take $q=1$, so that now

$$u_{-n}=-u_n,\qquad v_{-n}=v_n. \tag{8}$$

From the second of (8), it follows at once that $g_x\equiv v_x$ is not a solution of (5). If (A') were correct the contrary would be the case.

Let a, b, c, d, m, n, x, y denote real integers. Then $g_x \equiv u_x$ is a solution of (7), as is seen from (A), and hence also of (5), since $u_1 = 1$. To preserve the double homogeneity of (5) and its consequences we retain all powers of u_1 arising from special choices of a, b, c, d. Thus the formulas (B) should be written $(q = 1)$,

$$(B') \qquad \begin{aligned} u_n^3 u_{n+2} - u_{n+1}^3 u_{n-1} &= u_1^3 u_{2n+1}, \\ u_n^2 u_{n+1} u_{n+3} - u_n^2 u_{n-1} u_{n+1} &= u_1 u_2 u_{2n+2}. \end{aligned}$$

Write Jacobi's $\mathrm{H}(z) \equiv \mathrm{H}$, $\mathrm{H}(nz) \equiv \mathrm{H}_n$ $(n \neq 1)$. Then for $g_x \equiv \mathrm{H}_x$, (5) becomes a well known form of Jacobi's theta relation, so that H_x is a solution of (7). This verifies Lucas' statement that (A), (B) belong to the theory of the H function (with $q = 1$ and the above modification (B') of (B)). A somewhat closer analogy is given by the solutions

$$g_x \equiv \eta_x \equiv \mathrm{H}_x/\mathrm{H}^{x^2}, \qquad g_x \equiv \zeta_x \equiv \mathrm{H}_x \mathrm{H}^{(x^2-4)/4}/\mathrm{H}_2{}^{(x^2-1)/3}$$

of (5), ζ_x being the familiar elliptic function occurring in the theory of Poncelet polygons. In these cases, analogously to $(u_0, u_1) = (0, 1)$, we have

$$(\eta_0, \eta_1) = (\zeta_0, \zeta_1) = (0, 1).$$

Analogies with the Weierstrassian

$$\sigma(nz) \equiv \sigma_n, \quad \wp(nz) \equiv \wp_n$$

are obtained by means of the well known

$$\psi_n(z) \equiv \psi_n = \sigma_n/\sigma_1{}^{n^2},$$

which is fundamental in the theory of real multiplication. For $g_x \equiv \psi_x$ is a solution of (5), and $(\psi_0, \psi_1) \equiv (0, 1)$. Other solutions are Halphen's

$$\gamma_x \equiv \psi_x/\psi_2{}^{(x^2-1)/3}, \qquad \delta_x \equiv \gamma_{mx}\gamma_m{}^{(x^2-4)/3}/\gamma_{2m}{}^{(x^2-1)/3},$$

for which $(\gamma_0, \gamma_1, \gamma_2) = (\delta_0, \delta_1, \delta_2) = (0, 1, 1)$. In transposing formulas relating to γ, δ into terms of u, all powers of δ_2, γ_2

must be retained, since $u_2 = 1$ (when $q = 1$) only when $p = 1$. The solution $g_x \equiv \sigma_x$ of (5), giving Weierstrass' equation of three terms, yields analogies between u_x and σ_x. The function γ_x is that occurring in the conversion of the square root of a polynomial of the third or fourth degree into a continued fraction; δ_x is used in abridging the computation of γ_{mn}. In this last there is a resemblance to the process by which Lucas converts his u_x, v_x formulas into others concerning u_{mx}, v_{mx}. The usual methods for calculating γ_x as a polynomial in $\gamma_3{}^3$, γ_4 can evidently be carried over bodily to the computation of u_x. More generally, we see from the foregoing solutions of (5) that any relation between any one of H_x, η_x, ζ_x, σ_x, ψ_x, γ_x, δ_x for different ranks x can be transposed into a relation between u_x's of the same ranks. The converse does not hold; only such properties of u_x are translatable into terms of H_x, $\cdots$, δ_x as can be obtained from (5) alone. A correspondence between $\wp$'s and u's is established by means of the functions $-\wp_{x,\,y}$, $u_{x,\,y}$ of two arguments x, y (of which y does not assume the value zero),

$$\wp_{x,\,y} \equiv \wp_x - \wp_y = \wp_{x,\,1} - \wp_{y,\,1} = -\frac{\psi_{x+y}\psi_{x-y}}{\psi_x{}^2\psi_y{}^2},$$

$$u_{x,\,y} \equiv \frac{u_{x+y}u_{x-y}}{u_x{}^2u_y{}^2} = u_{x,\,1} - u_{y,\,1}.$$

Although Lucas did not consider u_x when x is a complex number, we see at once that (5) is valid when $q = 1$, a, b, c, d are complex, and $g_x \equiv u_x$. It has recently been shown by Berwick * that the same relation (5) holds for certain functions ψ upon which the complex multiplication of $\wp(z)$ depends. Thus u_x has analogies in both the real and the complex multiplication of elliptic functions.

THE UNIVERSITY OF WASHINGTON

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* PROCEEDINGS OF THE LONDON SOCIETY, vol. 19 (1920), p. 153.

SINGULARITIES OF CURVES OF GIVEN ORDER*

BY T. R. HOLLCROFT

1. *Introduction.* The following problems concerning plane curves are suggested by the theory of surfaces and space curves.

I. To find the greatest number of cusps that may be added to the singularities of a plane curve of order n with κ_1 cusps and δ_1 nodes.

II. To find the greatest number of nodes that may be added to the singularities of a plane curve of order n with κ_1 cusps and δ_1 nodes.

III. To find the greatest number of cusps that may occur among the remaining singularities of a plane curve of order n and genus p with κ_1 cusps and δ_1 nodes.

IV. To find the least order n of a plane curve that can possess a given number κ_1 of cusps and δ_1 of nodes.

These problems are all closely related to the question of the existence of plane curves with assigned singularities, in which the principal problem consists of finding the maximum number of cusps of a curve of given order and genus. This problem has been solved by Lefschetz,† who also mentions the first two of the above problems, but suggests only graphical solutions for them. The existence of curves with cusps up to and including the maximum has been shown by both Lefschetz and Coolidge ‡ for $p \leqq p_0$, where p_0 is the genus associated with a given n on the assumption that the bitangents and inflections are both as near zero as possible.

As a notation for the number of singularities of any algebraic plane curve of order n, class m, and genus p, we shall say that the curve has κ cusps, δ nodes, ι inflections, and τ bitangents. We shall have occasion to use the following formulas, the first three of which are Plückerian:

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* Presented to the Society, April 28, 1923.

† S. Lefschetz, *On the existence of loci with given singularities*, Transactions of this Society, vol. 14 (1913), pp. 23–41.

‡ J. L. Coolidge, *On the existence of curves with assigned singularities*, this Bulletin, vol. 28 (1922), pp. 451–455.

$$(1)\quad m = n(n-1) - 2\delta - 3\kappa,$$

$$(2)\quad \iota = 3n(n-2) - 6\delta - 8\kappa,$$

$$(3)\quad 2\tau = 2\delta + (m-n)(m+n-9),$$

$$(4)\quad F = \tfrac{1}{2}n(n+3) - \delta - 2\kappa - 8 = 3n + p - 9 - \kappa, \qquad p \geqq 2.$$

The last formula, due to Professor Lefschetz,* gives the number of independent absolute invariants necessary to determine a curve of order n and genus $p \geqq 2$ with δ nodes and κ cusps.

The conditions for the maximum number of cusps of a curve of given order and genus are obtained by Lefschetz for certain limits of n in terms of a given p. If we restate them for certain limits of p in terms of a given n, we find that for $p \leqq p_0$, where p_0 is defined by

$$[p_0]\dagger \leqq \tfrac{1}{2}(n + 2 - \sqrt{4n+13}),$$

κ is a maximum when ι is a minimum; for $p_0 < p \leqq p_1$, where

$$[p_1] \leqq \tfrac{1}{2}(2n - 1 + \sqrt{16n - 23}),$$

κ is a maximum when τ is a minimum; and for

$$p_1 \leqq p \leqq \tfrac{1}{4}(n-4)(n-5) \qquad \text{and} \qquad n > 13,$$

κ is limited by the number of independent absolute invariants determining the curve.

In all four problems, let κ_1 and δ_1 represent, respectively, the number of cusps and nodes already possessed by the curve of order n and let κ_m represent the maximum number of cusps that a curve of order n with δ_1 nodes may have. The solutions will now be given.

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* S. Lefschetz, loc. cit., p. 29.

† In this paper, the symbol $[x]$ followed by $\leqq$ ($\geqq$) means the largest (one greater than the largest if not equal) integer contained in the expression on the right. The interpretation not in parentheses is the usual one.

2. *Problem* I. From formula (2) we obtain, by means of the condition $\iota \geqq 0$,

$$[\kappa_m] \leqq \tfrac{3}{8}(n^2 - 2n - 2\delta_1). \tag{5}$$

Substituting the value of m from (1) in (3), letting $\tau \geqq 0$, and solving for κ, we obtain

$$[\kappa_m] \geqq \tfrac{1}{6}[2n(n-1) - 4\delta_1 - 9 - \sqrt{4n(n-9) - 8\delta_1 + 81}]. \tag{6}$$

If we let $F \geqq 0$ in (4), we find

$$[\kappa_m] \leqq \tfrac{1}{4}[n(n+3) - 2(\delta_1 + 8)]. \tag{7}$$

For a given n and δ_1, the one of these three formulas that gives the least value of κ_m determines the maximum number of cusps that curve can have.

It is now necessary to find the limits of δ_1 for which each of these formulas applies. If

$$\delta_1 \geqq \tfrac{1}{2}(n-2)(n-4),$$

the curve may be rational and have all the rest of its double points cusps. The limit (5) holds for $0 \leqq p \leqq p_0$ and for these curves

$$[\delta] \leqq \tfrac{1}{2}(n-2)(n-4) - 4p.$$

Replacing δ by δ_1, and eliminating p from this formula and the upper limit for p, we obtain

$$\delta_1 \geqq \tfrac{1}{2}n\,(n-10) + 2\sqrt{4n+13}.$$

To find the lower limit of δ_1 for formula (6) when $n > 13$, eliminate p from the inequality for p_1 and the formula for δ, when $p \geqq p_1$,

$$\delta = \tfrac{1}{2}(n-4)(n-5) - 2p.$$

This gives

$$\delta_1 \geqq \tfrac{1}{2}(n-2)(n-11) - \sqrt{16n-23}.$$

The complete solution may now be stated as follows:

If

$$\delta_1 \geqq \tfrac{1}{2}(n-2)(n-4),$$
$$\kappa_m = \tfrac{1}{2}(n-1)(n-2) - \delta_1.$$

If

$$\tfrac{1}{2}(n-2)(n-4) \geqq \delta_1 \geqq \tfrac{1}{2}n(n-10) + 2\sqrt{4n+13},$$
$$[\kappa_m] \leqq \tfrac{3}{8}(n^2 - 2n - 2\delta_1).$$

If

$$\tfrac{1}{2}n(n-10)+2\sqrt{4n+13} > \delta_1 \begin{cases} \geqq 0 & \text{for} \quad n \leqq 13,\\ \geqq \tfrac{1}{2}(n-2)(n-11) \\ \qquad - \sqrt{16n-23} & \text{for} \quad n > 13, \end{cases}$$

$$[\kappa_m] \geqq \tfrac{1}{6}[2n(n-1)-4\delta_1-9-\sqrt{4n(n-9)-8\delta_1+81}].$$

If

$$\tfrac{1}{2}(n-2)(n-11)-\sqrt{16n-23} \geqq \delta_1 \geqq 0 \quad \text{for} \quad n > 13,$$

$$[\kappa_m] \leqq \tfrac{1}{4}[n(n+3)-2(\delta_1+8)].$$

In any of these cases, the number of cusps that may be added is $\kappa_m - \kappa_1$, and there may be one additional node if κ_m is not diminished by substituting $\delta_1 + 1$ for δ_1 in the limit for κ_m.

3. *Problem* II. Let δ_m denote the greatest number of nodes (including δ_1) that the curve of order n with κ_1 cusps can have. Solve the three formulas (5), (6), (7) for δ in terms of n and κ_1 and find the limits of κ_1 for which they hold in the same way that the limits for δ_1 were found in the preceding problem In this case there can be no additional cusps. The results are as follows: If $\kappa_1 \leqq \frac{3}{2}(n-2)$, we have

$$\delta_m = \tfrac{1}{2}(n-1)(n-2)-\kappa_1.$$

If

$$\tfrac{3}{2}(n-2) \leqq \kappa_1 \leqq \tfrac{3}{2}(2n-\sqrt{4n+13}),$$

$$[\delta_m] \leqq \tfrac{1}{6}[3n(n-2)-8\kappa_1].$$

If

$$\tfrac{3}{2}(2n-\sqrt{4n+13}) < \kappa_1 \begin{cases} \leqq \tfrac{1}{3}n(n-2) & \text{for} \quad n \leqq 13,\\ \leqq \tfrac{1}{2}(8n-19+\sqrt{16n-23}) & \text{for} \quad n > 13, \end{cases}$$

$$[\delta_m] \leqq \tfrac{1}{2}(n^2-n-3\kappa_1-5-\sqrt{3\kappa_1-8n+25}).$$

If

$$\tfrac{1}{2}(8n-19+\sqrt{16n-23}) \leqq \kappa_1 \leqq \tfrac{1}{4}(n^2+3n-16) \qquad \text{for} \quad n > 13,$$

$$[\delta_m] \leqq \tfrac{1}{2}(n^2+3n-16-4\kappa_1).$$

In any case, the number of nodes that may be added is given by the difference $\delta_m - \delta_1$.

4. *Problem* III. In this problem p, κ_1, and δ_1 must satisfy the inequality

$$\tfrac{1}{2}(n-1)(n-2)-\kappa_1-\delta_1 \geqq p \geqq \tfrac{1}{2}(n-1)(n-2)-\kappa_1-\delta_m.$$

The problem is solved by noting that the maximum number of cusps depends only on n and p when

$$\delta_1 \leqq \tfrac{1}{2}(n-1)(n-2)-\kappa_m-p,$$

and when δ_1 equals or exceeds this limit, all the remaining double points may be cusps. The following formulas result: If

$$0 \leqq p \leqq \tfrac{1}{2}(n+2-\sqrt{4n+13}),$$

and

$$\delta_1 \leqq \tfrac{1}{2}(n-2)(n-4)-4p,$$
$$[\kappa_m] \leqq \tfrac{3}{2}(n+2p-2).$$

If

$$\tfrac{1}{2}(n+2-\sqrt{4n+13}) < p \leqq \tfrac{1}{2}(2n-1+\sqrt{16n-23})$$

and

$$\delta_1 \leqq \tfrac{1}{2}(n^2-7n+13-6p+\sqrt{24p-8n+25}),$$
$$[\kappa_m] \leqq 2(n+p)-\tfrac{1}{2}(11+\sqrt{24p-8n+25}).$$

If

$$\tfrac{1}{2}(2n-1+\sqrt{16n-23}) \leqq p \leqq \tfrac{1}{4}(n-4)(n-5)$$

and

$$\delta_1 \leqq \tfrac{1}{2}(n-4)(n-5)-2p,$$
$$\kappa_m = 3n+p-9.$$

If

$$p > \tfrac{1}{4}(n-4)(n-5),$$

or if δ_1 equals or exceeds any of the above limits,

$$\kappa_m = \tfrac{1}{2}(n-1)(n-2)-\delta_1-p.$$

In any case, the number of cusps that may be added is $\kappa_m-\kappa_1$, and the additional nodes, if any, that belong to the curve, are given by the formula

$$\delta = \tfrac{1}{2}(n-1)(n-2)-\kappa_m-\delta_1-p.$$

The solution of the problem, *to find the maximum number of nodes that may occur among the remaining singularities of a curve of order* n *and genus* p *with* κ_1 *cusps and* δ_1 *nodes*, is evidently always

$$\delta_m - \delta_1 = \tfrac{1}{2}(n-1)(n-2) - \kappa_1 - \delta_1 - p,$$

since p cannot be less than the genus of a curve with κ_1 cusps and the maximum number of nodes.

5. *Problem* IV. For a rational curve we have

$$\delta_1 \geqq \tfrac{1}{2}(n-2)(n-4), \tag{8}$$

$$\kappa_1 = \tfrac{1}{2}(n-1)(n-2) - \delta_1. \tag{9}$$

Solving (9) for n, we have

$$n = \tfrac{1}{2}[3 + \sqrt{8(\kappa_1 + \delta_1) + 1}].$$

Substituting this value of n in (8), and solving for δ_1, we obtain

$$\delta_1 \geqq \tfrac{2}{9}\kappa_1(\kappa_1 - 3),$$

which gives the relation between δ_1 and κ_1 for which the above formula holds.

When $\delta_1 < 2\kappa_1(\kappa_1 - 3)/9$ the curve cannot be rational and we must consider three cases, $0 \leqq p \leqq p_0$, $p_0 < p \leqq p_1$ and $p \geqq p_1$. For $p \leqq p_0$ the maximum number of cusps occurs when the number of inflections is a minimum. Solving the inequality resulting from equation (2) for n, we find

$$[n] \geqq \tfrac{1}{3}[3 + \sqrt{3(3 + 6\delta_1 + 8\kappa_1)}].$$

Eliminating p and n from the inequality defining p_0 and the two inequalities

$$[\delta_1] \geqq \tfrac{1}{2}(n-2)(n-4) - 4p,$$

$$[\kappa_1] \leqq \tfrac{3}{2}(n + 2p - 2),$$

which give the nodes and the maximum number of cusps for $p \leqq p_0$, we obtain

$$\delta_1 \geqq \tfrac{1}{18}[\kappa_1(\kappa_1 - 21) + 2\kappa_1\sqrt{3(\kappa_1 + 3)}].$$

If $p_0 < p \leqq p_1$, the maximum number of cusps occurs when the number of double tangents is a minimum. If we attempt to solve the resulting inequalities in κ_1, δ_1 and n for n, we are led to the quartic equation

$$n^4 - 2n^3 - [2(3\kappa_1 + 2\delta_1) + 9]n^2 + 2(3\kappa_1 + 2\delta_1 + 9)n \\ + (3\kappa_1 + 2\delta_1)(3\kappa_1 + 2\delta_1 + 9) + 2\delta_1 = 0.$$

A general solution of this equation is too involved to be of use as a formula, but the value of n for given values of κ_1 and δ_1 can be found as the least positive real root of this quartic when that root is an integer, or the integer next larger when the least positive root is irrational.

If

$$\tfrac{1}{18}[\kappa_1(\kappa_1 - 21) + 2\kappa_1\sqrt{3(\kappa_1 + 3)}] > \delta_1 \geqq 0,$$

and $\kappa_1 \leqq 50$ the foregoing formula applies. If $\kappa_1 > 50$, however, the lower limit for δ_1, for which the foregoing formula applies, is not zero, but we cannot find it in general, since it involves the solution of two quartic equations.

If $\kappa_1 > 50$, and δ_1 is less than this lower limit, $p > p_1$, and the least value of n is obtained by eliminating p from the two formulas

$$\kappa_1 = 3n + p - 9,$$
$$p = \tfrac{1}{2}(n - 1)(n - 2) - \delta_1 - \kappa_1.$$

Solving these for n, we have

$$[n] \geqq \tfrac{1}{2}(\sqrt{16\kappa_1 + 8\delta_1 + 73} - 3).$$

The inequality sign is introduced in the result because δ_1 may not have the value that will satisfy the equality.

In general, when it is desired to find the least order n of a curve that can possess a given number κ_1 of cusps and δ_1 of nodes, we may proceed as follows. If

$$\delta_1 \geqq \tfrac{2}{9}\kappa_1(\kappa_1 - 3),$$
$$[n] = \tfrac{1}{2}[3 + \sqrt{8(\kappa_1 + \delta_1) + 1}].$$

If

$$\tfrac{2}{9}\kappa_1(\kappa_1 - 3) > \delta_1 \geqq \tfrac{1}{18}[\kappa_1(\kappa_1 - 21) + 2\kappa_1\sqrt{3(\kappa_1 + 3)}],$$
$$[n] \geqq \tfrac{1}{3}[3 + \sqrt{3(3 + 6\delta_1 + 8\kappa_1)}].$$

If

$$\tfrac{1}{18}[\kappa_1(\kappa_1 - 21) + 2\kappa_1\sqrt{3(\kappa_1 + 3)}] > \delta_1 \geqq 0 \quad \text{and} \quad \kappa_1 \leqq 50,$$

n is given as the least positive real root, if that is an integer, or the integer next larger than the least positive irrational root of the quartic

$$n^4 - 2n^3 - [2(3\kappa_1 + 2\delta_1) + 9]n^2 + 2(3\kappa_1 + 2\delta_1 + 9)n + (3\kappa_1 + 2\delta_1)(3\kappa_1 + 2\delta_1 + 9) + 2\delta_1 = 0.$$

If $\kappa_1 > 50$, determine n by the inequality

$$[n] \geqq \tfrac{1}{2}(\sqrt{16\kappa_1 + 8\delta_1 + 73} - 3).$$

If $p \geqq p_1$ for this value of n and the given values of κ_1 and δ_1, this is the correct value for n; but if $p < p_1$, this formula does not apply, and n must be determined from the quartic equation as above.

When $\delta_1 = 0$, the solution is more simple. For $n \leqq 13$, the maximum number of cusps is given by $[\kappa] \leqq \frac{1}{3}n(n - 2)$. Hence, solving this inequality for n, we have

$$[n] \geqq 1 + \sqrt{3\kappa_1 + 1} \qquad \text{for} \qquad \kappa_1 \leqq 50.$$

For $n > 13$,

$$[\kappa] \leqq \tfrac{1}{4}n(n + 3) - 4.$$

Solving for n we obtain

$$[n] \geqq \tfrac{1}{2}(\sqrt{16\kappa_1 + 73} - 3) \qquad \text{for} \qquad \kappa_1 > 50.$$

In the foregoing results, nothing has been said as to the relative positions of the double points. None of the curves can have more double points than has been assigned them, but in certain cases some of these double points may unite to form points of higher multiplicity. When the genus does not change, one additional independent absolute invariant is imposed on the curve for each increase in the multiplicity of a point by unity. For these curves, the only restriction on the coincidence of double points is that the total increase in multiplicity shall not exceed $3n + p - 9 - \kappa$. For $p < p_1$, this allows a certain number of coincidences, but for $p \geqq p_1$, all the double points assigned must be distinct for a proper curve.

Wells College

THE HURWITZ–COURANT FUNKTIONENTHEORIE

Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen. Von Adolf Hurwitz. Herausgegeben und ergänzt durch einen Abschnitt über *geometrische Funktionentheorie*, von R. Courant. Berlin, Julius Springer, 1922. vi + 399 pp.

The present volume is the third of a series * of mathematical monographs under the editorship of R. Courant, which was inaugurated in 1921 with the first volume of Blaschke's *Vorlesungen über Differentialgeometrie*. The announcements of the series and the initial volume have been such as to lead one to expect thoroughly modern and otherwise valuable treatments of the subjects considered, and it is therefore with disappointment that one examines the present volume, in spite of certain merits which it has.

According to the preface, the first two parts are the lecture notes of Hurwitz, almost unchanged, written from the Weierstrassian standpoint. The third part is by Courant, and is based on the ideas of Riemann. Thus one is prepared for differences, but hardly such striking differences as are found. In fact, so lacking is organic connection, so dissimilar in spirit are the works of the two authors and so different their qualifications, that one can only regard the volume as two books printed as one.

Part I consists of 131 pages, devoted to the general theory of functions, the power series being the basis. Cauchy's integral theorem is introduced and used effectively, somewhat beyond the middle of the treatment—it might have been employed earlier with profit. The ground covered is surprisingly extensive; this is partly due to the fact that a knowledge of the foundations of the theory of functions of a real variable is presupposed, and partly to a rare skill in the arrangement, and a willingness on the part of the author to forego extreme generality in his theorems. The style is lucid and the reviewer noticed no faults in the logic. A sketch of the contents follows: complex numbers and their representations on plane and sphere; power series, their circle of convergence, reckoning with them, their identical vanishing, their continuation, their differentiation and integration; the notion of analytic function; the elementary functions; integration, the theory of residues, Laurent series, and related questions; meromorphic functions, the Mittag-Leffler theorem on the partial fraction development of a meromorphic function, the factorization of transcendental integral functions; the inversion of power series and the Lagrange series.

Part II comprises 112 pages on elliptic functions, also lucidly written, and abounding in valuable results and formulas. The transition from Part I is made naturally, by a consideration of the properties of the general periodic meromorphic function. The salient topics which follow are: the Weierstrass function $\wp(u)$, the invariants connected with it, the expression of the general elliptic function in terms of $\wp(u)$ and its derivative; the function $\sigma(u)$; the theta functions; the functions sn u, cn u, and dn u; the elliptic modular functions; the uniformization of certain algebraic relationships; the elliptic integrals; the Landen transformation.

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* *Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete.*

In marked contrast to the first two parts, Part III (149 pages) aims at a high degree of generality. It gives the impression of being the work of a mind endowed with fine intuitive faculties, but lacking in the self-discipline and critical sense which beget confidence. The progress in geometric function theory of the last two decades, and the need of a systematic exposition of the results, have been such as to lead to the hope for an adequate presentation here. What is found may, indeed, serve as an indication of some of the directions which modern investigations have taken—in fact, a very interesting one. But the proofs offered often leave the reader unconvinced as to their validity and, at times, uncertain as to whether they can be made valid.

The initial paragraphs of Part III are devoted to line integrals, Green's theorem, and the mapping properties of the general analytic function. A stationary flow of an incompressible fluid is defined, according to the author, by a vector field, without restriction. The integral over a closed curve of the normal component of the field is declared to represent the diminution of the quantity of the fluid (Flüssigkeitsmenge) in the region bounded by the curve per unit of time! (One is tempted to ask, first, whether *quantity* means volume, or mass; and, secondly, in what sense the quantity of fluid within the circle $x^2 + y^2 = 1$, whose velocity field is given by $u = x$, $v = y$, diminishes by 2π units per second.) This may serve as an example of the author's interest in the "besondere Berücksichtigung der Anwendungsgebiete" of the announcement of the series. A second treatment of Cauchy's integral theorem, and of the logarithmic function exhibits a lack of correlation with the first part of the book.

The boundary value problem of potential theory for the circle is treated by Poisson's integral and a Cesàro summation. There follows an acceptable discussion of the mapping properties of the elementary functions, of the elliptic integral of the first kind, a study of the process of analytic continuation by reflection, of the functions connected with the triangle with straight or circular segments as sides, the differential equation satisfied by the functions connected with polygons of circular arcs, and Picard's theorem. The theorem of Darboux that if $\zeta = f(z)$ is regular within a region B bounded by a regular curve C, and continuous in the closed region, and if $\zeta = f(z)$ maps C on a curve C' without double points, then this function maps the interior of C on the interior of C' in a one-to-one manner, is left half proved. The statements (pp. 299–300) as to the generalizations of the principle of analytic continuation by reflection are, to say the least, insufficiently grounded.

The concluding chapter constitutes the central, and most interesting, portion of Part III. The fundamental mapping theorem is attacked in the form: the general simply or multiply connected open continuum can always be mapped onto a "Schlitzbereich," i.e., the whole plane bounded by certain straight line segments all lying on a set of parallels.* The proof of the theorem, to which 26 pages are devoted, is, in the nature of things, difficult. It is here attacked by the method of minimizing a Dirichlet integral. While there are a number of places where statements are unsubstantiated, the impression is left that the methods are in general direct and well chosen, and that a revision of the treatment so as to complete a real proof would be a possible and worth while labor. It is rather in the extensions that more serious doubts assail one.

The later sections of the chapter take up abelian integrals and algebraic functions on Riemann surfaces, the general problem of uniformization, and the conformal mapping of multiply connected surfaces on each other.

The reader who wishes an orientation with respect to geometric function theory, and who will not be misled as to validity of proofs, will find Part III interesting, and even stimulating. There are excellent possibilities ahead for a revision. The proof reading has been well done, and the typography and general appearance of the book are models of excellence.

O. D. KELLOGG

CORRECTIONS

BY HAROLD HILTON

In the article on pages 303–308 of the July issue of this BULLETIN (vol. 29, No. 7) the following corrections by the author were received by the editors after the final proofs had been returned to the printers.

On page 304, in line 13, change $(p - q)/q$ to $(p - q)/p$.

On page 304, in line 23, change ϕ to ω.

On page 305, in line 19, change $K\pi\epsilon$ to $K\pi/\epsilon$; and change $e^{-k\pi t/\epsilon} \sin k\pi\epsilon$ to $e^{-K\pi t/\epsilon} \sin (k\pi/\epsilon)$.

On page 305, in line 3 from the bottom, change T to τ.

On page 308, omit entirely lines 10–13.

On page 308, in line 3 from the bottom, change (3), (4) to (8), (9).

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* The open continuum in question is supposed to be of finite connectivity in the proof offered. Later it is asserted that the theorem admits of generalization to the case of infinite connectivity, but we are not told what becomes of the Schlitzbereich.

SHORTER NOTICES

Vorlesungen über die Grundzüge der mathematischen Statistik. 2d edition. By C. V. L. Charlier. Lund, Verlag Scientia. 125 pp.

During the past two decades there has been a considerable advancement in the mathematics of statistics by workers in the countries of northern Europe. The results of their activity have been rather slow in coming to the attention of Americans because most of the work has been published in one of the Scandinavian languages or at least in Scandinavian journals. Of these workers, Charlier is one of the best known. In his published memoirs he goes back to pre-Gaussian times and builds up a logical science of statistics from a few principles from Laplace, making much use of the theory of the superposition of small errors.

The little book under review sums up the results of his work. It is not a treatise or a text-book but simply a book of directions for applying his methods to statistical data with many problems worked out in detail. For mathematical details the reader is referred to the original articles. The book begins with the usual discussion of the arithmetic mean, measures of dispersion and probable error, using Charlier's own self-checking plan of computation. Then follows a very clear discussion, well illustrated by examples, of the series of Bernoulli, Poisson and Lexis leading to the notions of "übernormal" and "unternormal" dispersion of Lexis and to Charlier's "coefficient of disturbancy," a measure of the effect of causes which cannot be explained by the theory of probability.

To American readers the most interesting part of the book is that dealing with Charlier's two representations of frequency curves. Type I curves are represented by

$$f(x) = \beta_0 \varphi_0(x) + \beta_3 \varphi_0'''(x) + \beta_4 \varphi_0^{\mathrm{IV}}(x) + \cdots,$$

where

$$\varphi_0(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}$$

and the prime marks denote differentiation. Type II curves are given by

$$F(x) = N[\psi(x) + \gamma_2 \Delta^2 \psi + \gamma_3 \Delta^3 \psi + \cdots],$$

where

$$\psi(x) = \frac{e^{-\lambda} \lambda^x}{x!} \qquad \text{and} \qquad \Delta\psi = \psi(x) - \psi(x-1).$$

In the text Charlier outlines plans of the numerical work for calculating the constants β, λ, and γ, and carries out the computations for two frequency distributions. One of the most interesting questions in mathematical statistical circles at present is the analytical representation of frequency distributions. In this country and in England Pearson's methods have long held sway, and now comes a rival, Charlier. It seems logical to think that Charlier's methods should give a better fit than the more empirical and pragmatic methods of Pearson, but actual tests do not always bear this out. Experience alone will tell us which method is best in actual service.

The last three chapters in the book are devoted to the theory of correlation, first the general theory and then that for four-fold tables. Chapter XIII is a tantalizing chapter. It begins with the statement that two phenomena are correlated, when in whole or in part they are the resultants of the same elementary causes, and then simply gives the ordinary product-moment formula for the correlation coefficient with plans for calculating the various constants.

Interested students will find Charlier's methods discussed in considerable detail in Arne Fisher's recent work, *The Mathematical Theory of Probability*.

A. R. Crathorne

Théorie Mathématique des Phénomènes Thermiques produits par la Radiation Solaire. By M. Milankovitch. Paris, Gauthier-Villars, 1920. xvi + 340 pp.

This book is of great interest in several respects. It is written by a Serb and was interrupted by war when the author was taken prisoner by Austria-Hungary. Granted the freedom to pursue his work the book was completed owing to the courtesy of the Hungarian Academy of Sciences. But its greater title to interest is that, apparently, it is the first complete treatise on the thermal effects of solar radiation treated from a systematic point of view.

At the first stage the problem is considered of the amount of heat received per unit area on a rotating planet devoid of atmosphere with a sun fixed in distance and direction. Account is then taken of the elliptic orbit and obliquity of the ecliptic to determine the "radiation-constant" day by day. In particular, the amount of heat received in each season of the year is expressed by formula.

Up to this point the work is practically a branch of celestial kinematics. At the next step the loss of radiation due to absorption by an atmosphere etc. is considered, especially as it tends to diminish the intensity of oblique rays. This factor has a large influence on the temperature of points near the arctic circle. It is due to this factor that a variation in the obliquity of the ecliptic may produce large changes in the climate of higher latitudes.

Passing over an excursion relative to the effect of conduction of heat in the solid crust, the next topic is the determination of the loss caused by the atmosphere. This loss consists partly of scattering in the air, and reflection by clouds and by the surface of the earth, partly by true absorption with re-emission at a much longer wave-length. One outstanding fact is that the atmosphere absorbs far more of the outwardly emitted long-wave radiation than of the incoming visible light.

To give a mathematical theory it is necessary to consider the thermal mechanics of gases. Two simplifying assumptions may be made. Of these, adiabatic equilibrium is shown to lead to results quite out of keeping with observation. On the other hand, the assumption of temperature equilibrium under absorption and emission of radiation appears to fit the facts well, and when there are two gases with distinct properties a reversal of temperature gradient is shown theoretically to be possible. A particular example is given by a mixture of oxygen and carbon dioxide such as that actually in the atmosphere.

Finally the theory is applied to the special case of the earth and, also, to the three other minor planets, and for the earth numerical tables are given. The question of glacial periods is handled in a moderate spirit, critically and without controversial prejudice. Since glaciation is not caused by cold winters (cf. Siberia) but by long and cold summers (that is, by cool periods of precipitation) the effect on temperate climates of a variation in the obliquity of the ecliptic combined with a high eccentricity of orbit may be quite large.

Stockwell's formula for the variations of e and ϵ are used to calculate the amount of heat received each season and the length of each season for the 500 millennia before 1850. This formula is of so complicated and empirical a form that one can have little confidence in the extreme degree of extrapolation which such a use implies. It would seem to be enough, however, to warrant the conclusion that glacial epochs could be explained if the variations have been in the past slightly greater than those given by the exact formula.

The author is to be warmly congratulated on such a thorough and critical attack on the problem of "mathematical climate." The book can be recommended, not only to meteorologists, but also to those who enjoy the clearing up of new fields in physics by the use of straightforward mathematics with numerical determinations. It would exert a broadening influence on a student of mathematical physics in whose hands it might be placed, since it combines problems from several distinct fields.

P. J. Daniell

Géométrie Descriptive. By Gaspard Monge. Augmentée d'une *Théorie des Ombres et de la Perspective,* extraite des papiers de l'auteur par Barnabé Brisson. Parts I and II. Paris, Gauthier-Villars, 1922. xvi + 144 + 138 pp.

These reprints, appearing in the series Les Maitres de la Pensée Scientifique ably edited by Maurice Solovine, are of special interest to Americans for the reason that the descriptive geometry created by Monge was among the earliest mathematical creations of the French which reached the United States; it was taught at West Point in the first quarter of the nineteenth century. Monge's descriptive geometry was first published in the year VII (1799), but the present reprint is from the fourth (1820) posthumous edition which included also material on shadows and perspective, selected by Brisson from Monge's then unpublished papers. Of interest is Monge's estimate of the value of descriptive geometry: "If in all the larger towns there were established secondary schools, in which young men of the age of twelve years who look forward to practicing some one of the trades should be trained during two years in graphic constructions and become acquainted with the principal phenomena of nature, a knowledge of which is indispensable to them, this training, in developing their intelligence and giving them the habit and the sentiment of precision, would contribute in the surest manner to the progress of national industry."

Florian Cajori

Differential- und Integralrechnung. Band I. *Differentialrechnung.* By Ludwig Bieberbach. Leipzig and Berlin, B. G. Teubner, 1922. vi + 131 pp.

This little book is one of the series known as Teubners Technische Leitfaden. It is indicated in the announcement of this series that the purpose is to give to the reader, in brief and practical yet accurate form, the most essential points of the subjects treated. The book under review seems to accomplish this purpose admirably for the subject "of differential calculus."

The book contains the following chapters: I, Der Funktionsbegriff; II, Der Zahlbegriff; III, Unendliche Reihen; IV, Stetige Funktionen; V, Differentialrechnung; VI, Einige geometrische Anwendungen; VII, Die Taylorsche Formel; VIII, Unbestimmte Formen; IX, Beispiel einer stetigen nirgend differenzierbaren Funktion; X, Funktionen von zwei Variablen.

The topics thus indicated (together with the integral calculus) are about the same as those sometimes given to students in this country under the title, Second Calculus. The treatment, however, is quite brief, and somewhat more informal than usual. Many will no doubt regard this as a distinct merit.

The very critical can easily raise objections here and there. Some will think it particularly disadvantageous to regard $\sqrt{x}$ as a double valued function, as the author does on page 1. Again some will wonder if they are to infer, from the definition of continuous function (page 56), that a continuous function must be single valued. If so, the statement on page 61, line 3, that $\sqrt[n]{x}$ is continuous would imply that this function is also single valued, contrary to the statement at foot of page 1. But if this is not intended, then the $f(x)$ in Rolle's Theorem (page 82) should be required to be *eindeutig*. There are not many, however, who will be misled by small faults of this character, and in the main, the reasoning is clear and accurate. The literary style is, throughout, particularly pleasing for a work on mathematics.

There are occasional misprints. In line 19, p. 52, occurs Inter*f*allanfang for Inter*v*allanfang, and in line 4 of Satz 2, page 58, is found $f(x) = m$, instead of $f(x) - m$. The reference to page 8 in the index after the words "maximum" and "minimum" also seems erroneous.

LOUIS INGOLD

Kraftarten und Bewegungsformen. By Max Möller. Braunschweig, Friedrich Vieweg und Sohn, 1922. viii + 148 pp.

This little book contains a classification and study of the various kinds of mechanical force, work, impulse, and energy. The theory is presented in clear and simple form with numerous applications to gases, liquids, and solids. Many of the examples are carried through to a numerical conclusion with a careful statement of the units in which the various quantities are expressed. It should prove useful in connection with courses in mechanics which are often presented in too abstract a fashion to give perfectly clear and concrete notions of the fundamental quantities involved.

H. B. PHILLIPS

Frequency Arrays. By H. E. Soper. Cambridge, University Press, 1922. 48 pp.

This booklet gives a very condensed treatise upon the advantages to be gained in the use of "symbols bearing an objective or logical significance" in the study of statistical and other distributions. No attempt has been made to establish any new results, but any knowledge of the mass of results treated here, and the fact that the treatment is restricted to less than fifty pages, should produce faith in anyone in the power and possible conciseness of the method of attack recommended by the author.

No one is advised to try to read this booklet who is not already familiar with most of the work and results of Pearson and his disciples, for it contents itself for the most part with the very concise reproduction of most of the most important results of Pearson and others, with practically no explanation of those results. This is no criticism, of course, because the book is designed for the use of the advanced student who is interested in research in statistics and who desires a very concise method of attack. The book then will be of little or no use in any undergraduate course in statistics, even as a reference book. For this reason it would perhaps be well for the reviewer to limit further account to a mere list of some of the most important topics included:

Moment arrays.

Binomial, Poisson, Gaussian, exponential and gamma types of frequencies; including the simple, multiple and general Gaussian distributions, the correlation surface, multiple correlation, Gaussian derivatives, Thiele derivatives, tetrachoric functions.

Sampling without replacement and partitioning a limited population.

Hypergeometric and kindred frequencies; including moments of hypergeometrical frequencies and of the double hypergeometrical series of frequencies.

Geometrical distributions; Samples of vectors; Random migration; including polar symbols in two dimensional distributions, circular distributions and vector sampling, moments of tensors of circular distributions, the symmetrical Gaussian, circular area, uneven distribution of points on a circle, single points, non-circular distributions and vector sampling, polar symbols in three dimensional distributions, spherical distributions and vector sampling, fortuities of resultant tensor when n randomly selected tensors are set in random directions in a space of s dimensions.

C. H. Forsyth

Atomes et Électrons. Institut International de Physique Solvay. Rapports et discussions du Conseil de Physique tenu à Bruxelles du 1er au 6 avril 1921. Paris, Gauthier-Villars, 1923. 8 + 273 pp.

The International Institute of Physics was founded in 1912 by Ernest Solvay for the object of encouraging physical research. One of the means of accomplishing this object was the organization of international congresses, with a limited number of participants, meeting at Brussels from time to time.

This volume is the third that has been published, containing the papers read, and the discussions regarding them, at the Congress of 1921. The two former volumes were entitled *La Théorie des Rayonnements et les Quanta*, published in 1912, and *La Structure de la Matière*, published in 1921. From the titles of these volumes it will be seen that the three congresses that have already been held have been concerned with questions that are of the greatest interest in the present state of physical science.

The volume under review begins with a paper on the *Theory of electrons*, by Lorentz, and includes a discussion of effects due to the rotation of electrons. Sir Ernest Rutherford reports on the structure of atoms with particular reference to the interpretation of his experiments on atomic disintegration as a result of collisions between atoms and rapidly moving α-particles. The experimental results, which, at present, can be interpreted only with the aid of the hypothesis of energy quanta, $h\nu$, are described by de Broglie. Kamerlingh Onnes has two reports; one on paramagnetism at low temperatures, considered from the point of view of the constitution of the elementary magnets, and a second report on the superconducting state of metals, considered with reference to the Rutherford-Bohr atomic model. Sir W. H. Bragg reports on the result of measurements of the intensity of Röntgen rays reflected from diamond. Experiments on the angular momentum accompanying magnetization, in iron and nickel, which result in observing only half the effect to be expected if the elementary magnets are electrons circulating in closed orbits, are described by de Haas. Finally, there are two papers on the quantum theory; one by Bohr on its application to atomic problems, and another by Ehrenfest on Bohr's principle of correspondence.

A valuable feature of this volume, as well as the preceding ones, is the discussions following the various reports. These discussions, sometimes raising additional difficulties, and sometimes bringing out alternative views and new results, serve to show, not only how far from finality is the solution of these fundamental problems, but also the immense progress that has been made in recent years in proposing such problems for solution.

E. P. Adams

Bessel Functions. By A. Gray, G. B. Mathews and T. M. MacRobert. London, Macmillan, 1922. xiv + 327 pp.

This is a second edition of the classical treatise of Gray and Mathews prepared by Andrew Gray and T. M. MacRobert. The general plan of the original treatise has been retained but many changes of detail have been made. The analytic work at the beginning of the book has been rewritten and a collection of examples has been added to each of the first seven chapters. More than half of the book is devoted to physical applications, the topics treated including vibrating membranes, hydrodynamics, steady flow of electricity and heat, propagation of electro-magnetic waves, and diffraction. At the end of the book is a collection of tables and a bibliography of the more important treatises and memoirs.

H. B. Phillips

Geschichte der Mathematik. By H. Wieleitner. I. *Von den ältesten Zeiten bis zur Wende des* 17. *Jahrhunderts.* Berlin, Vereinigung Wissenschaftlicher Verleger, 1922. 136 pp.

The author of this remarkable little compendium has distinguished himself by numerous valuable contributions to the history of mathematics. This work justifies the expectation of excellence inspired by the demonstrated ability of the writer. Particularly valuable is the summary on Arabic achievements (pp. 44–55) and equally valuable the discussion of the Latin Middle-Ages, commonly neglected.

Up to the time of printing the material is treated by countries; the modern period is subdivided into the development of the older mathematics and the birth of modern mathematics. The first of these sections is discussed under the four topics: the spread of mathematical interest, the development of algebra, of geometry, and of trigonometry. The second section takes up the following topics: the ideas culminating in the infinitesimal calculus, the discovery of differential and integral calculus, the origin of analytic geometry, the beginning of projective geometry, the algebra of modern times, and further development of trigonometry.

Highly commendable is the wise selection and the equitable distribution of the material. The only serious criticism which can be brought is a tendency to minimize the Hindu achievements, giving credit to hypothetical Greek works not yet discovered.

For a rapid survey of the history of mathematics there is no better little treatise than this modest volume, priced at 25 cents, bound.

L. C. KARPINSKI

Introduction à la Géométrie Non-Euclidienne. By A. MacLeod. Paris, J. Hermann, 1922. 433 pp.

The plan of this elementary book is similar to that of the first seven chapters of J. L. Coolidge's text (*The Elements of Non-Euclidean Geometry*, Oxford, 1909). The author has followed that text rather closely in some places, but in other places he disagrees with the proofs and substitutes his own. He presents his material in a much less condensed form and adds a chapter on n-dimensional geometry (a subject which J. L. Coolidge had purposely omitted).

The book is gratefully dedicated to Professor G. Mittag-Leffler on the occasion of his seventy-fifth anniversary.

E. B. COWLEY

Nichteuklidische Geometrie. By Heinrich Liebmann. Dritte, neubearbeitete Auflage. Berlin and Leipzig, Walter de Gruyter, 1923. 150 pp.

The first edition of this book was reviewed in this BULLETIN (vol. 13 (1906–7), pp. 511–12). In this third edition the general plan of the earlier editions has been retained but the material has been further developed, new drawings have been made, and references have been added to books which have been published more recently. Thus the valuable little book has been brought up to date.

E. B. COWLEY

NOTES

At the twentieth western meeting of this Society to be held on December 28–29, in Cincinnati, (see p. 197, May, 1923) the headquarters for mathematicians will be the Hotel Sinton, where rooms may be secured in advance at rates from $3.00 up.

The San Francisco Section of this Society will hold a special meeting at the University of Washington, in Seattle, on Saturday, December 22. The first session will be in Philosophy Hall, at 10 a. m.

The opening number of volume 25 of the TRANSACTIONS OF THIS SOCIETY (vol. 25, No. 1, Jan., 1923), which has appeared recently, was printed in Hamburg, Germany, by the firm of Lütcke and Wulff, on account of the continued difficulties connected with the printing of scientific journals in this country.

The opening number of volume 24, series 2, of the ANNALS OF MATHEMATICS contains: *Periodicities in the theory of partitions*, by E. T. Bell; *Functionals of summable functions*, by W. L. Hart; *Periodically closed chains of reduced fractions*, by A. Arwin; *On certain linear differential equations of the second order*, by F. H. Murray.

The Class of Sciences of the Royal Academy of Belgium announces the following subjects for prizes to be awarded in 1924: (1) *a contribution to the infinitesimal geometry of curved surfaces*; (2) *a new contribution to the question of the absorption of light in interstellar space*. Competing memoirs should be presented by August 1, 1924.

The Class of Physical Sciences of the Academy of Bologna announces the following subjects for a prize to be awarded in 1925: (1) *a contribution to the theory of the uniformization of an algebraic equation* $f(x, y) = 0$ *(the expression of* x *and* y *in uniform functions of a parameter), or an exposition of the present status of the question*; (2) *a discussion of some important subject in the history of mathematical analysis in the eighteenth century*. If no suitable memoirs are presented on these subjects, the Academy may consider any unpublished work in mathematical analysis. The competition closes December 31, 1924.

The Italian Society of Sciences has awarded its gold medal for 1922 to Professor Gaetano Scorza, of the University of Naples, for his book, *Corpi numerici ed algebre*, and its gold medal for 1923 to Professor Leonida Tonelli for volume I of the treatise *Fondamenti di Calcolo delle Variazioni*.

At the meeting of the Société Mathématique de France on July 11, papers were read by Professor Edward Kasner, on *Ricci curvature and its generalization*, and by Professor J. F. Ritt, on *The algebraic functions which can be expressed by means of radicals*.

Professor Jean Perrin, professor of physical chemistry at the Sorbonne, has been elected a member of the Paris Academy of Sciences, in the section of physics.

Professor H. Villat, of the University of Strasbourg, has been elected a correspondent in the section of mechanics of the Paris Academy of Sciences.

Professor Tullio Levi-Civita has been elected a foreign honorary member of the Royal Society of Edinburgh.

Professor U. Cisotti, of the Technical School at Milan, has been elected a corresponding member of the Reale Academia dei Lincei, and Professor A. E. H. Love of Oxford University a foreign member.

Professor Niels Bohr has received the honorary degree of doctor of science from Cambridge University, and has been elected an honorary member of the Cambridge Philosophical Society.

Cambridge University has conferred the doctorate of science on Professor H. A. Lorentz, of the University of Leiden.

The University of Munich has conferred an honorary doctorate on Professor F. von Lindemann, on the occasion of his seventieth birthday.

At a meeting of the American Philosophical Society on April 10, the James Scott medal was awarded to Sir Joseph Thomson, for his work on the physics of the electron.

The University of St. Andrews has conferred an honorary doctorate on Dr. H. W. Richmond, of Cambridge University, retiring president of the London Mathematical Society.

The *Tables for Interior Ballistics*, which were issued some time ago in blue print form, have now been published in printed form, "For official use only," as "Ordnance Document, No. 2369." These tables were computed in the Ballistic Section of the Ordnance Department, under direction of Professor A. A. Bennett. They are the first tables of the sort available for the U. S. Army, and are made possible by the use of certain mathematical transformations discovered by Professor Bennett. They render obsolete the method of individual trajectory computations for interior ballistics, which has been hitherto the only theoretically accurate procedure.

The firm of Julius Springer, of Berlin, has just announced that it will allow a discount of 25 per cent to members of the American Mathematical Society and to members of the Mathematical Association of America on individual orders of copies of the MATHEMATISCHE ANNALEN, of the MATHEMATISCHE ZEITSCHRIFT, and of volumes of the series edited by R. Courant under the general series title *Die Grundlehren der Mathematischen Wissenschaften*. This discount is based on the Auslandspreis, which is $6.40 per volume for each of the journals, so that the net cost to a member of either organization would be $4.80. The prices of volumes of the Courant series vary, of course, but they will not exceed one cent per page for bound volumes.

At the recent meeting of the Deutsche Mathematiker Vereinigung in Marburg, the yearly dues of that organization were fixed at one-half gold mark. The charge for the JAHRESBERICHT remains at two dollars for members.

At the University of Paris, Dr. Gaston Julia has been promoted to a professorship of general mathematics.

Professor M. Cipolla, of the University of Catania, has been called to the chair of higher analysis in the University of Palermo.

Mr. G. C. Steward, of Gonville and Caius College, has been appointed fellow and lecturer in mathematics at Emmanuel College, Cambridge.

Dr. John Marshall, of University College, Swansea, has been appointed reader in mathematics at Bedford College, University of London.

Mr. R. Stoneley has been appointed assistant lecturer in applied mathematics at the University of Leeds.

Professor C. E. Cullis, of the University of Calcutta, has retired from the Indian Educational Service, and returned to England.

The following have been appointed privat docents: at the University of Genoa, Dr. F. Sbrana, in rational mechanics; at the University of Naples, Dr. S. Cherubino, in calculus, and Dr. M. Pascal, in rational mechanics; at the University of Pisa, Dr. G. Albanese, in analytic geometry, Dr. M. Bedarida, in algebraic analysis, and Dr. F. Cecioni, in calculus; at the University of Münster, Dr. M. Krafft; at the Hanover Technical School, Dr. A. Schur.

Dr. V. D. Gokhale, of the University of Chicago, has been appointed associate professor of mathematics at the University of the Philippines.

Professor E. A. Kholodovsky, assistant professor of mathematics at the Polytechnic Institute of Petrograd, has been appointed assistant in the Lick Observatory.

Professor W. B. Fite, of Columbia University, Treasurer of the American Mathematical Society, has been granted leave of absence for the first half of the academic year 1923–24.

Professor B. H. Camp, of Wesleyan University, during his leave of absence for the academic year 1923–24, plans to study at the University of Paris and with Professor Karl Pearson at the Biometric Laboratory of University College, London.

Professor W. H. Kirchner, of the department of drawing and descriptive geometry at the University of Minnesota, has sabbatical leave of absence for the year 1923–1924, and expects to spend much of his time studying at the University of Palermo.

Professor Archibald Henderson, of the University of North Carolina, has been granted leave of absence on the Kenan Research Foundation for the academic year 1923–1924, and will carry on in Europe researches in the theory of relativity.

Professor W. H. Metzler, of Syracuse University, has been appointed dean of the New York State College for Teachers at Albany.

At the University of West Virginia, Dr. B. M. Turner, of the University of Illinois, has been appointed assistant professor of mathematics.

Dr. F. W. Reed, of Cornell University, has been appointed assistant professor of mathematics at Ohio State University.

Dr. George Rutledge, of the Massachusetts Institute of Technology, has been promoted to an assistant professorship of mathematics.

Assistant Professor J. E. Davis, of the University of Arkansas, has been appointed assistant professor of mathematics at Drexel Institute.

Mr. H. T. Davis, of the University of Wisconsin, has been appointed assistant professor of mathematics at the University of Indiana.

Miss Lena R. Cole has been made head of the department of mathematics at Central Normal College, Danville, Ind.

Dr. Nina M. Alderton has been promoted to be head of the department of mathematics in Mills College.

Professor B. R. Allen, of Westmoorland College, has been appointed associate professor of mathematics in the Kansas State Teachers College at Emporia.

Professor A. A. McSweeny, of the College of Agriculture and Mechanic Arts of the University of Montana, has been appointed head of the department of mathematics at John Tarleton Agricultural College, Stephensville, Texas.

Professor W. W. Weber, dean and professor of mathematics at Southern College, Lakeland, Florida, has been appointed to the chair of mathematics in Lander College, Greenwood, S. C.

Professor S. I. Jones, of Lipscomb College, has become assistant treasurer of the Life and Casualty Insurance Company of Nashville.

At Heidelberg University, Professor J. E. Pierce of James Millikin University has been made head of the department of mathematics.

The following appointments to instructorships in mathematics in American colleges and universities are announced:

Cornell University, Mr. B. F. Kimball;
Heidelberg University, Mr. C. E. Stout;
University of Michigan, Mr. B. F. Dostal;
Southwestern Presbyterian College, Mr. M. L. MacQueen;
University of West Virginia, Dr. M. M. Feldstein;
University of Wyoming, Miss Minnie Holman.

Mr. W. E. Armentrout, who had been instructor in mathematics at the University of Wisconsin, and was under appointment as instructor at Cornell University for the year 1923–1924, was struck by lightning June 18, 1923, at Madison, Wis., and was instantly killed.

Professor A. D. Pitcher, of Western Reserve University, died suddenly from heart failure at his home in Cleveland, on October 5, 1923, at the age of forty-three years. He had been a member of this Society since 1910, and was, at the time of his death, an elected member of its Council.

NEW PUBLICATIONS

I. PURE MATHEMATICS

BLASCHKE (W.). Vorlesungen über Differentialgeometrie. Band 2: Affine Differentialgeometrie, bearbeitet von K. Reidemeister. Berlin, 1923. 10 + 259 pp.

BOREL (E.) et DELTHEIL (R.). Probabilités, erreurs. Paris, Armand Colin, 1923. 16mo. 200 pp.

VAN DER CORPUT (J. G.). Grepen uit de getallenleer. Rede uitgesproken bij de aanvaarding van het hoogleeraarsambt aan Rijksuniversiteit Groningen op zaterdag 17 maart 1923. Groningen, Wolters, 1923.

DELTHEIL (R.). See BOREL (E.).

GRÜNBAUM (M.). Teoremas de geometria. Barlad, Sociedad Matematica Barladeana, 1923. 16 pp.

HELSINGFORS. Wissenschaftliche Vorträge gehalten auf dem fünften Kongress der Skandinavischen Mathematiker in Helsingfors vom 4. bis 7. Juli 1922. Helsingfors, Akademische Buchhandlung, 1923. 4 + 315 pp.

KIEPERT (L.). Grundriss der Differential-Rechnung. 14te, vollständig umgearbeitete und vermehrte Auflage. Band II. Hannover, Helwingsche Verlagsbuchhandlung, 1923.

LIEBMANN (H.). Nichteuklidische Geometrie. 3te, neubearbeitete Auflage. Berlin, Vereinigung wissenschaftlicher Verleger, 1923. 150 pp.

MINEUR (A.). Cubiques anallagmatiques. Bruxelles, J. van Dijl, 1923. Lithographed. 77 pp.

NAGEL (T.). Sur la distribution des nombres qui sont premiers avec un nombre entier donné. Christiania, Morten Johansen, 1922. 36 pp.

PRASAD (G.). Mathematical research in the last 20 years. Presidential address delivered on the 31st January 1921 before the Benares Mathematical Society. Berlin, Vereinigung wissenschaftlicher Verleger, 1923. 36 pp.

REIDEMEISTER (K.). See BLASCHKE (W.).

VON SANDEN (H.). Praktische Analysis. (Handbuch der angewandten Mathematik, herausgegeben von H. E. Timerding, 1ter Teil.) 2te, verbesserte Auflage. Leipzig, Teubner, 1923. 18 + 195 pp.

SMAIL (L. L.). Elements of the theory of infinite processes. New York, McGraw-Hill, 1923. 336 pp.

TIMERDING (H. E.). See VON SANDEN (H.).

VAULOT (A.). Congruences rectilignes qui sont en même temps W et de Ribaucour. (Thèse, Paris.) Paris, Gauthier-Villars, 1923. 95 pp.

WEITZENBÖCK (K.). Invariantentheorie. Groningen, Nordhoff, 1923.

WEYL (H.). Mathematische Analyse des Raumproblems. Berlin, Springer, 1923. 8 + 118 pp.

ZIPPERER (L.). Tafeln zur harmonischen Analyse periodischer Kurven. Berlin, Springer, 1922. 9 Abbildungen und 23 graphischen Berechnungstafeln.

II. APPLIED MATHEMATICS

d'Adhémar (R.). Statique, cinématique. (Eléments de mécanique à l'usage des ingénieurs.) Paris, Gauthier-Villars, 1923. 11+254 pp.

Alliatta (G.). Die neueste Orientierung der Physik. 1: Sinn und Bedeutung des Michelsonschen Versuches. 2: Zur Theorie der Elektronenröhre. Leipzig, Hillmann, 1922.

Amaldi (U.). See Levi-Civita (T.).

Becquerel (J.). See Metz (A.).

Bellenot (H.). See Sommerfeld (A.).

Bergson (H.). Durée et simultanéité; à propos de la théorie d'Einstein. 2e édition, augmentée. Paris, Alcan, 1923. 10 + 289 pp.

Blair (E. W.). See Born (M.).

Bloch (E.). Les phénomènes thermioniques. Paris, Les Presses Universitaires de France, 1923. 111 pp.

Bohr (N.). Ueber die Quantentheorie der Linienspektren. Uebersetzt von P. Hertz. Braunschweig, Vieweg, 1923. 4 + 168 pp.

Born (M.). The constitution of matter. Modern atomic and electron theories. Translated from the second revised German edition by E. W. Blair and T. S. Wheeler. London, Methuen, 1923. 7 + 80 pp.

Calderwood (J.). See Thorne (A. T.).

Carr (H. W.). The general principle of relativity in its philosophical and historical aspect. 2d edition, revised and enlarged. London, Macmillan, 1922. 8 + 200 pp.

Corps (—.). Les théories de la relativité dépassent les données de l'expérience. Paris, Gauthier-Villars, 1923. 4to. 43 pp.

Cutter (L. E.). Descriptive geometry. New York, 1923. 244 pp.

Czuber (E.). Mathematische Bevölkerungstheorie auf Grund von G. H. Knibb's "The mathematical theory of population." Leipzig, Teubner, 1923. 16 + 357 pp.

Dacrement (E.). Electricité. Partie I: Théorie, production, transformation. 2e édition, mise à jour par L. Grininger. Paris, Dunod, 1923. 12 + 846 pp.

Einstein (A.). The meaning of relativity. Princeton, N. J., Princeton University Press, 1923. 123 pp.

Encyklopädie der mathematischen Wissenschaften. Band VI 2 B, Heft 1: Oppenheim, Die Theorie der Gleichgewichtsfiguren der Himmelskörper; Kritik des Newtonschen Gravitationsgesetzes, mit einem Beitrag von Kottler, Gravitation und Relativitätstheorie. Leipzig, Teubner, 1922.

Erasmus Bartholinus. Versuche mit dem doppelbrechenden Isländischen Kristall, die zur Entdeckung einer wunderbaren und aussergewöhnlichen Brechung führten. (Ostwald's Klassiker der Exakten Wissenschaften, Nr. 205.) Leipzig, Akademische Verlagsgesellschaft, 1922.

de Fontviolant (B.). Résistance des matériaux analytique et graphique. Paris, Baillière, 1923. 580 pp.

Fuchs (R.) und Hopf (L.). Aerodynamik. Berlin, R. C. Schmidt, 1922.

GANDILLOT (M.). L'étherique. Essai de physique expérimentale. Paris, Vuibert, 1923. 4 + 950 pp.

GLAZEBROOK (R.). Dictionary of applied physics. Volume 5: Aeronautics, metallurgy, general index. London, Macmillan, 1923. 7 + 592 pp.

GRAETZ (L.). Handbuch der Elektrizität und des Magnetismus. Band II: Elektronen und Ionen. Leipzig, Barth, 1923. 12 + 1074 pp.

GRININGER (L.). See DACREMONT (E.).

HAY (A.). Alternating currents; their theory, generation, and transformation. 5th edition, revised and enlarged. London, Harper, 1923. 436 pp.

HENDERSON (A.). Relativity. A romance of science. Chapel Hill, N. C., University of North Carolina Press, 1923. 66 pp.

HERTZ (P.). See BOHR (N.).

HOPF (L.). See FUCHS (R.).

DE JANS (C.). Sur le mouvement d'une particule matérielle dans un champ de gravitation à symétrie sphérique. Paris, Gauthier-Villars, 1923. 98 pp.

JEANS (J. H.). The nebular hypothesis and modern cosmogony. Oxford, Clarendon Press, 1923. 8vo. 31 pp.

KIRCHBERGER (P.). Was kann man ohne Mathematik von der Relativitätstheorie verstehen? 3te, vermehrte und verbesserte Auflage. Karlsruhe, C. F. Müller, 1922. 8 + 95 pp.

KNIBBS (G. H.). See CZUBER (E.).

KOPFF (A.). The mathematical theory of relativity. Translated by H. Levy. London, Methuen, 1923. 8 + 214 pp.

KOTTLER (—.). See ENCYKLOPÄDIE.

KRUPPA (E.). See MÜLLER (E.).

VON LAUE (M.). Das physikalische Weltbild. Vortrag, gehalten auf der Kieler Herbstwoche 1921. Karlsruhe, C. F. Müller, 1921. 25 pp.

LEVI-CIVITA (T.) e AMALDI (U.). Lezioni di meccanica razionale. Volume 1: Cinematica, principi e statica. Bologna, Zanichelli, 1923.

LEVY (H.). See KOPFF (A.).

MAGNEL (G.). Pratique du calcul du béton armé. Partie 1. Paris, Gauthier-Villars, 1923. 160 pp.

MAILLARD (L.). Quand la lumière fut . . . Tome 2: Les cosmogonies modernes. Paris, Les Presses Universitaires de France, 1923. 280 pp.

MATISSE (G.) See SILBERSTEIN (L.)·

METZ (A.). La relativité. Préface de J. Becquerel. Paris, E. Chiron, 1923. 156 pp.

MÜLLER (E.). Vorlesungen über darstellende Geometrie. Band 1: Die linearen Abbildungen bearbeitet von Dr. E. Kruppa. Leipzig, Deuticke, 1923. 292 pp.

NERNST (W.) und SCHOENFLIES (A.). Einführung in die mathematische Behandlung der Naturwissenschaften. 10te, vermehrte und verbesserte Auflage. München und Berlin, Oldenbourg, 1923. 6 + 502 pp.

OBERTH (H.). Die Rakete zu den Planetenraumen. München und Berlin, Oldenbourg, 1923. 92 pp.

OPPENHEIM (—.). See ENCYKLOPÄDIE.

ORLANDI (G.). Nuove tavole tacheometriche per determinare le distanze orizzontali, le differenze di livello, le coordinate planimetriche e le curve. 3a edizione. (Manuali Hoepli.) Milano, Hoepli, 1922. 24 + 201 pp.

OSTERTAG (P.). Kolben- und Turbo-Kompressoren. Theorie und Konstruktion. 2te, verbesserte Auflage. Berlin, Springer, 1923.

ROSS (A. D.). A popular introduction to Einstein's theory of relativity, with an account of the tests made by the Wallal solar eclipse expedition. Perth, E. S. Wigg, 1923. 32 pp.

ROY (M.). Sur la théorie des surfaces portantes. (Collection Scientia.) Paris, Gauthier-Villars, 1923. 132 pp.

SCHMID (T.). Darstellende Geometrie. 2te Auflage. Band II. Berlin, Vereinigung wissenschaftlicher Verleger, 1923. 340 pp.

SCHOENFLIES (A.). See NERNST (W.).

SCHULTZE (A.). Ist die Welt vierdimensional? Leipzig, Hillmann, 1922.

SILBERSTEIN (L.). Eléments de la théorie électromagnétique de la lumière. Traduit de l'anglais par G. Matisse. Paris, Gauthier-Villars, 1923. 8vo. 4 + 94 pp.

SOMMERFELD (A.). La constitution de l'atome et les raies spectrales, traduite sur la 3e édition allemande par H. Bellenot. 2e fascicule. Paris, A. Blanchard, 1923. 350 pp.

SOUTHALL (J. P. C.). Mirrors, prisms, and lenses. A text-book of geometrical optics. Enlarged and revised edition. New York, Macmillan, 1923. 20 + 657 pp.

STRASSER (H.). Einsteins spezielle Relativitätstheorie. Eine Komödie der Irrungen. Bern und Leipzig, E. Bircher, 1923. 59 pp.

THORNE (A. T.) and CALDERWOOD (J.). Notes on torsional oscillations with special reference to marine reduction gearing. London, Spon, 1923. 51 pp.

UNITED STATES NAVAL ACADEMY. Elementary mechanics, prepared by the department of mathematics of the United States Naval Academy. Revised. Annapolis, 1922. 352 pp.

URBAN (F. M.). Grundlagen der Wahrscheinlichkeitsrechnung und der Theorie der Beobachtungsfehler. Leipzig, Teubner, 1923. 6 + 274 pp.

VASILIEFF (A. W.). Space, time, motion. Introduction to general relativity. (In Russian.) Berlin, Argonautenverlag, 1922. 156 pp.

VILAR (A.). Notes sur les distances des planètes au soleil. Paris, Jouve et Cie., 1923. 76 pp.

DE VRIES (H.). Leerboek der beschrijvende meetkunde. Deel II: Ruimtekrommen en gebogen oppervlakken. Tweede druk. Delft, J. Waltman, 1922.

WAGNER (P.). Der Energiebegriff. Entwurf zur Erkenntnisgrundlage der Ursachen aller Erscheinungen. Charlottenburg, Selbstverlag, 1922. 96 pp.

WHEELER (T. S.). See BORN (M.).

THE THIRTIETH SUMMER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY

The thirtieth summer meeting of the Society was held at Vassar College, Poughkeepsie, New York, on Thursday and Friday, September 6–7, 1923, the college opening its buildings for the entertainment of the visitors and thus contributing greatly to the success of the sessions. The Mathematical Association of America held its meetings immediately preceding and a joint session of the two societies was arranged for Thursday afternoon, at which addresses were delivered by Professor L. J. Mordell, of the University of Manchester, and Professor R. C. Archibald, of Brown University. At the joint dinner on Thursday evening, President McCracken of Vassar College welcomed the visiting mathematicians and spoke on the relation of the undergraduate college to research. Tea was served every afternoon after the sessions at the home of Professor and Mrs. H. S. White. At the last session, it was voted to express to Vassar College and to the department of mathematics the thanks of the Society for their generous hospitality and for the admirable arrangements.

The attendance included the following seventy-nine members of the Society:

Agard, Archibald, G. N. Armstrong, Babb, Bacon, Barney, E. R. Beckwith, S. R. Benedict, Bernstein, E. W. Brown, H. S. Brown, J. A. Bullard, W. G. Bullard, R. W. Burgess, Cairns, G. M. Conwell, Copeland, Court, L. D. Cummings, Dimick, Dines, Eiesland, Feldstein, W. B. Ford, Glenn, M. C. Graustein, W. C. Graustein, Grove, E. R. Hedrick, A. M. Howe, Huntington, Ingels, Kazarinoff, Kline, W. D. Lambert, F. P. Lewis, Luck, McCain, McDonnell, MacDuffee, Mathews, H. H. Mitchell, C. N. Moore, Mordell, Morenus, C. C. Morris, Nassau, Pierpont, Pitcher, Post, Rambo, Ranum, Reddick, L. J. Reed, R. G. D. Richardson, E. D. Roe, J. R. Roe, Rosenbach, Seely, Sinclair, Slaught, Slobin, C. E. Smith, D. E. Smith, Gertrude Smith, W. M. Smith, M. J. Sperry, B. M. Turner, Tyler, Vandiver, Veblen, M. E. Wells, H. S. White, Whittemore, Constance Wiener, Norbert Wiener, E. W. Wilson, Worthington, J. W. Young.

At the meeting of the Council, the following twenty-seven persons were elected to membership in the Society:

Professor Samuel Latimer Boothroyd, Cornell University;
Mr. Leonidas Hamlin Bunyan, University of Wisconsin;
Miss May Bryan Carter, Western College;
Professor Jacques Chapelon, University of Lille;
Rev. Gabriel Hippolytus Collignon, Loyola University;
Mr. Paul Sumner Dwyer, Pennsylvania State College;
Miss Lois Wilfred Griffiths, University of Washington;
Professor Lilian May Hackney, Marshall College;
Professor George Wellman Hess, Whitworth College;
Mr. Arthur Owen Hickson, Brown University;
Captain Grafton S. Kennedy, Ordnance Department, United States Army;
Mr. Harry Raymond Kimball, San Francisco, Calif.;
Mr. Harry Levy, Princeton University;
Dr. Elsie Jeannette McFarland, University of California;
Mr. Robert Houghton Marquis, Pennsylvania State College;
Professor Rhesa Lancaster Newlin, Guilford College;
Professor Niels Erik Nörlund, University of Copenhagen;
Mr. Howard Percy Robertson, University of Washington;
Mr. Henry Allen Robinson, Johns Hopkins University;
Professor Lao Genevra Simons, Hunter College;
Mr. Marshall Harvey Stone, Harvard University;
Mr. John Clement Tinner, Chicago, Ill.;
Lieutenant Lawrence Wainwright, United States Navy;
Professor Benjamin Lewis Waits, Alcorn College;
Professor Anne Marie Whelan, Olivet College;
Miss Evelyn Prescott Wiggin, Brown University;
Mr. Hugh Herbert Wolfenden, Grimsby, Ontario.

Forty-nine applications for membership were received. The Secretary announced that the following members of the London Mathematical Society had accepted membership under the reciprocity agreement since the April meeting.

Mr. Charles G. F. James, London;
Professor Louis Joel Mordell, University of Manchester.

The Council announced the appointment of the following committees: Professors Tyler, E. R. Hedrick, R. G. D. Richardson, Slaught, Veblen, H. S. White and J. W. Young on revision of the by-laws; Professors Eisenhart, Birkhoff and Dickson on nomination of officers; Professors Veblen, Birkhoff, Eisenhart, E. R. Hedrick, Slaught and J. W. Young on printing of the journals. Professor Birkhoff represented the Society at the inauguration of President Stratton of the Massachusetts Institute of Technology on June 11, 1923.

It was announced that the Guarantee Fund for the expenses of the endowment campaign had been raised and that the campaign itself would be formally opened in October. The whole-hearted cooperation of the members of the Society was invited. The committee on incorporation reported that it was confidently expected that the necessary legal forms could be attended to so that the incorporated body would supersede the unincorporated before the end of the year.

A resolution was adopted sanctioning the establishment of a lectureship to be known as the Josiah Willard Gibbs Lectureship, the lecture to deal in semi-popular form with some aspect of mathematics or its applications. The President was authorized to appoint a committee to make arrangements for the first lecture, which will probably be given in New York City during the winter of 1923–24.

In view of the anticipated meeting of the International Mathematical Congress in Canada in the summer of 1924 the Society decided to omit its summer meeting for that year.

At the meeting of the Society and the Association it was voted to request the Secretaries of the two organizations to send a letter to the Physico-Mathematical Society of Japan, expressing the sympathy of American mathematicians for their colleagues in Japan under the calamity that has befallen their country through the great earthquake.

At the joint session the following papers were read:

I. *An introductory account of the arithmetical theory of algebraic numbers and its recent development*, by Professor L. J. Mordell. (Address delivered at the request of the American Mathematical Society.)

II. *Mathematicians and music*, by Professor R. C. Archibald. (Address of the retiring President of the Mathematical Association of America.)

President Oswald Veblen presided at the regular sessions of the Society, relieved by Professor H. S. White. Titles and abstracts of the papers read at these sessions follow below. The papers of Professor R. L. Moore, Dr. Murray, Professor Altshiller-Court, Professor Murnaghan, Dr. Walsh, Mr. Thomas, Mr. McDonnell, Mr. Michal, and Professor Evans were read by title; Mr. Rainich was introduced by Professor Kasner, and Mr. Michal by Professor Evans.

1. Professor H. S. White: *Note on five points and a cyclic correspondence.*

Five given numbers in a given cyclic order determine completely a (2, 2) correspondence. The explicit equation is deduced, and an algebraic proof of its periodicity.

2. Professor B. A. Bernstein: *A generalization of the syllogism.*

This paper will appear in full in an early issue of this BULLETIN.

3. Professor B. A. Bernstein: *Operations with respect to which the elements of a boolean algebra form a group.*

Let a' denote the negative of a boolean element a, $a + b$ the logical sum of a and b, and ab their logical product; the author pointed out in a previous paper that $ab' + a'b$ and $ab + a'b'$ are operations with respect to each of which the elements of a boolean algebra form an abelian group. In this paper he determines all the operations of a boolean algebra with respect to which the elements form a group in general and an abelian group in particular.

4. Mr. H. S. Vandiver: *A new type of criteria for the first case of Fermat's last theorem.*

By transforming the criteria of Kummer, the author shows that if $x^p + y^p + z^p = 0$ is satisfied in integers x, y, and z, not zero and prime to the odd prime p, then

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \cdots + \frac{1}{u^2} \equiv 0 \pmod{p},$$

where u is the greatest integer in $p/3$. The paper will be offered for publication to the ANNALS OF MATHEMATICS.

5. Mr. H. S. Vandiver: *A method for finding a factor of an integer of the form* $8n + 1$.

If an integer m of the form $8n + 1$ is the product of two prime factors, then each factor has the same residue modulo 8. The author shows that m may be represented in at least two different ways in one of the forms $x^2 + y^2$, $x^2 + 2y^2$, $2x^2 - y^2$, where $x < \sqrt{p}$, $y < \sqrt{p}$. The factors of m are found from the different representations. The paper will be offered for publication to this BULLETIN.

6. Mr. G. Y. Rainich: *The distribution of primes and the finiteness of the number of discriminants with a given number of classes.*

In this paper a proof of the following proposition is given: the number of integers m for which the ratio of the whole number of primes between m and m^2 to the number of primes of the form $x^2 + xy + my^2$ between the same limits does not exceed a given value is finite. It seems probable that when the number of classes of the discriminant $1-4m$ is fixed the aforesaid ratio does not exceed a certain value (which is very near to the double of the number of classes); should this be proved, the finiteness of the number of discriminants with a given number of classes would be established. The degree of non-unicity of decomposition in prime factors is characterized by the maximum number of factors in a product which is divisible by a prime without any partial product being divisible by a prime; the connection of this constant in the case of quadratic fields with the structure of the group of classes is shown. It is proved that when the number of classes is unity all the numbers $x^2 + xy + my^2$ below m^2 are primes (for x and y relative primes).

7. Professor O. E. Glenn: *A complete system of differential parameters of orders < 3 of the binary differential cubic.*

This paper, combined with two communicated previously as preliminary reports (this BULLETIN, April and May, 1923), forms a memoir entitled *On the reduction of differential parameters in terms of finite sets, with remarks concerning differential invariants of analytic transformations,* which has been offered to the AMERICAN JOURNAL OF MATHEMATICS. The finite system determined for the cubic belongs to the domain of rational polynomials in derivatives of orders $\leqq 3$ of the functions $x_i = \varphi_i(y_1, y_2)$ $(i = 1, 2)$ of the transformations.

8. Professors E. R. Hedrick and Louis Ingold: *Analytic and non-analytic functions in three dimensions.*

In a paper by Hedrick, Ingold, and Westfall, soon to appear in the JOURNAL DE MATHÉMATIQUES, it is shown that analytic functions of a complex variable may be distinguished from non-analytic functions by the fact that the Tissot indicatrix becomes a circle in the analytic case; also by the fact that certain characteristic directions become indeterminate. In the present paper the authors show that for transformations of space into itself, there is an ellipsoid at each point which plays a rôle analogous to that of the Tissot indicatrix in the plane, and there are at each point characteristic directions. In particular they study those transformations for which this ellipsoid reduces to a sphere, and the characteristic directions become indeterminate. It is found that these transformations are analogous to analytic functions. The transformations coincide with conformal transformations of space, and the set of functions defining them satisfy equations analogous to the Cauchy-Riemann equations; also each of these functions satisfies an equation analogous to Laplace's equation.

9. Professor R. L. Moore: *A connected and regular point set which contains no arc.*

It is shown that there exists a connected point set which is regular (connected im kleinen) at every one of its points but which contains no simple continuous arc.

10. Professor R. L. Moore: *Concerning the sum of a countable infinity of continua in the plane.*

Sierpinski has shown that there exists no bounded plane continuum which is the sum of a countable infinity of mutually exclusive closed point sets. He has raised the question whether this result would still hold good if the word "bounded" were omitted. In the present paper it is shown that, for the special case where the closed point sets in question are all continua, this question may be answered in the affirmative.

11. Professor R. L. Moore: *A continuum considered as the sum of its prime elements.*

Hans Hahn has recently introduced the notion of the prime elements (Primteile) of a continuum and has proved a theorem which may be interpreted to mean that, with respect to its prime elements considered as points, every continuum which is irreducible between two points is a simple continuous arc which has those points as end points. In the present paper it is shown, among other things, that every bounded continuum whatsoever is a continuous curve with respect to its prime elements considered as points.

12. Professor Mary E. Sinclair: *The brachistochrone with variable end points.*

Conditions for fixed end points require that the extremals be single arches of inverted cycloids. The first derivatives of the extremal-integral, $J(u, v)$, where u and v are the parameters of two fixed plane curves on which the end points vary, gives the known transversality conditions. Its second derivatives give conditions involving the curvatures of the fixed curves and the parameters of the cycloid at the end points, such that if but one end point vary, its focal point on the arch may be studied. If both end points vary, the second derivatives give inequalities which may be interpreted as restricting the curvature of one of the fixed curves when that of the other and the parameters of the cycloid are given. A geometric interpretation of these results is given. Sufficient conditions for a relative minimum follow from the theorem of Hahn used by Merrill in his generalization of the problem of Dido.

13. Professor R. G. D. Richardson: *A new necessary condition for relative extrema in quadratic and hermitian forms.*

If $A = \sum_{ij}^{1,n} a_{ij}x_jx_i$, $B = \sum_{ij}^{1,n} b_{ij}x_jx_i$ be real quadratic forms of which B is positive definite, the minimum of A subject to the condition $B = 1$ must be one of the n solutions $x_1^{(i)}, \cdots, x_n^{(i)}$ of the equations $\sum_j (a_{ij} - \lambda b_{ij}) x_j = 0$, corresponding to one of the n zeros λ_i of the determinant $|a_{ij} - \lambda b_{ij}|$. The new necessary condition for a minimum obtained by the author determines the particular solution furnishing the minimum. For the λ_i which is desired, it is necessary that in the determinant $|a_{ij} - \lambda_i b_{ij}|$ the sequence σ_i $(i = 0, \cdots, n)$ of sums of all the principal minors obtained by deleting i rows and the same i columns show no permanence of sign. This condition together with those already known forms a sufficient set.

More generally, when the linear conditions $\sum_{ij} b_{ij} x_j^{(1)} x_i = 0, \cdots, \sum_{ij} b_{ij} x_j^{(m)} x_i = 0$ are added to the problem, the minimum is again a solution of the same linear equations and a similar new necessary condition on minors of the bordered determinant completes the set of sufficient conditions.

The same theory applies also in the case of hermitian forms.

14. Professor C. N. Moore: *On the summation of trigonometric series by Euler's method.*

In this paper it is shown that while Euler's method of summation will serve to sum certain divergent trigonometric series, it is less effective than Cesàro's method when applied to Fourier's series. For it follows from a general criterion due to Lebesgue that Euler's method will not serve to sum at points of continuity of the function developed all the Fourier's series corresponding to functions having a Lebesgue integral, whereas it is a well known theorem that Cesàro's method has this property.

15. Professor L. L. Dines: *Concerning a suggested and discarded generalization of the Weierstrass factorization theorem.*

This paper will appear in full in an early issue of this BULLETIN.

16. Professor L. L. Dines: *A theorem on the factorization of polynomials of a certain type.*

Polynomials of the form $P \equiv y^n + P_{n-1}(x)y^{n-1} + \cdots + P_1(x)y + P_0(x)$, where the coefficients $P_j(x)$ are power series converging in a sufficiently restricted neighborhood of the origin and vanishing with x, are of importance in the theory of implicit functions. In the present paper, the author gives a direct algebraic proof of the following algebraic theorem relative to such polynomials: *If the characteristic line (Newton polygon) of the polynomial P is a broken line of k segments, then P is a product of k polynomials of the same type as P, each of which has an unbroken characteristic line the same in length and direction as one segment of the characteristic line of P.* The theorem is for this type of polynomials a converse of the "Theorem on the Product" proved by Blumberg (TRANSACTIONS OF THIS SOCIETY, vol. 17, p. 530) for a very general class of polynomials defined postulationally.

17. Mr. Donat Kazarinoff: *The scientific work of A. M. Liapounoff.*

This paper gives an account of the scientific work of the eminent Russian mathematician A. M. Liapounoff (born in Yaroslavl, 1857; died in Odessa, 1918). His achievements won wide recognition; he was a member of the Russian Academy of Sciences, a foreign member of the Accademia dei Lincei, a correspondent of the Académie des Sciences, etc. His basic contributions are on the figures of equilibrium of a homogeneous rotating liquid.

The present paper depends for its information almost entirely upon the obituary memoir by V. A. Stekloff, BULLETIN OF THE RUSSIAN ACADEMY OF SCIENCES, ser. 6, vol. 13 (1919), pp. 367–388.

18. Dr. Norbert Wiener: *The quadratic variation of a function.*

In this paper, a discussion of the second degree analog of the total variation of a function is employed to establish certain inequalities connecting the total variation of a function, its saltus, and its Fourier coefficients.

19. Dr. F. H. Murray: *Certain orbits with arbitrary masses in the problem of three bodies.*

This paper is devoted chiefly to the study of certain special cases of the problem of three bodies, by methods in which the notion of "system of invariant relations" as defined by Poincaré plays an important rôle. After certain properties of differential systems which admit a set of invariant relations have been developed, some qualitative results are obtained concerning isosceles triangle solutions with axis of symmetry. The equations of the problem of three bodies in the plane are given in a comparatively simple form, with the aid of which solutions asymptotic to the straight line and equilateral triangle solutions, respectively, are obtained. These results complete, on certain points, the researches of D. Buchanan concerning these solutions.

20. Professor W. C. Graustein: *Applicability with preservation of both curvatures.*

This paper will appear in full in an early issue of this BULLETIN.

21. Professor W. C. Graustein: *Isometric W-surfaces.*

This paper contains a complete classification of isometric W-surfaces and a discussion of the new types found. Special attention is given to isometric W-surfaces admitting continuous deformations into themselves.

22. Mr. G. Y. Rainich: *A new kind of representation of curved space.*

Given a surface S and a plane P we draw through a point A of S the normal to S and the perpendicular to P. We call the intersection of the bisector of the angle formed by these two lines with P the generalized stereographic projection of A. Vectors belonging to S (cf. an article by the author in the PROCEEDINGS OF THE NATIONAL ACADEMY, June, 1923) can also be projected stereographically, and from this arises a new kind of representation of tensors belonging to the surface through tensors of a plane, which has an advantage over the usual representation so far as algebraic relations are concerned, since no such distinctions as that between covariant and contravariant quantities are introduced. Differentiation, on the other hand, is more complicated in this system. Projecting the neighborhood of a point stereographically on the tangent plane at this point, and using cartesian coordinates in this plane, we obtain "semi-cartesian" coordinates for the representation of the surface. We can use a similar representation for higher spaces and also for curves on a plane.

23. Professor Nathan Altshiller-Court: *On two circles.*

The pair of lines joining the two points of intersection of two circles to a variable point on one of these circles determine in the second circle a chord of constant length. This length x is given by the proportion $x : s = d : p$, where s, d, p are the lengths of the common chord, the line of centers, and the radius of the first circle respectively. The author applies this result to a coaxial system of circles, to various circles connected with the triangle, etc., and gives a projective generalization of the proposition.

24. Professor F. D. Murnaghan: *A minimum problem in elementary geometry.*

The problem of finding the point the sum of whose distances to four points in space is a minimum was proposed by Steiner and has been considered by many geometers, notably by R. Sturm. Although Steiner gave a hint of the proper mode of attack, his hint seems to have escaped attention. It is apparent that if planes each through one of the four points perpendicular to its join to a fifth point form an equifacial tetrahedron, this fifth point will make $r_1 \pm r_2 \pm r_3 \pm r_4$ a minimum or a maximum. To find such a fifth point x (using as coordinates perpendiculars on the faces of the tetrahedron formed by the four given points) note that the pedal tetrahedron of its isogonal conjugate $y = 1/x$ will also be equifacial, and hence have its opposite edges equal. There are therefore 8 points y, the base points of the net of quadrics $y_1^2 + y_2^2 + 2c_{12}y_1y_2 = y_3^2 + y_4^2 + 2c_{34}y_3y_4$, $y_1^2 + y_3^2 + 2c_{13}y_1y_3 = y_2^2 + y_4^2 + 2c_{24}y_2y_4$, $y_1^2 + y_4^2 + 2c_{14}y_1y_4 = y_2^2 + y_3^2 + 2c_{23}y_2y_3$, where c_{rs} is the cosine of the internal angle between the faces r and s of the tetrahedron of reference.

25. Dr. J. L. Walsh: *A generalization of evolutes.*

Let points P_1 and P_2 trace two curves C_1 and C_2 so that the tangents to C_1 and C_2 at P_1 and P_2 are parallel. Let the point P be determined so that it divides the segment P_1P_2 in the constant ratio $m_1 : m_2$; P traces a curve C. Then the centers of curvature Q_1, Q_2, Q of C_1, C_2, C at corresponding points P_1, P_2, P are such that Q divides the segment Q_1Q_2 in the ratio $m_1 : m_2$. The circles of curvature of C_1, C_2, C at P_1, P_2, P have as common center of similitude the point of tangency of the variable line P_1P_2 with its envelope. Consider the family of curves found by varying the ratio $m_1 : m_2$. An infinity of curves of this family can be traced by points of an elastic string which moves so as to wrap itself on the envelope of the line P_1P_2.

26. Mr. J. M. Thomas: *Congruences of circles studied with reference to the surface of centers.*

The surface of centers S, rather than the envelope of the planes, is used as the framework of reference. Certain theorems relating to S in general are first proved. The main part of the paper is devoted to congruences C characterized as follows: each point P of a circle of the congruence is on a surface whose tangent plane at P contains the radius to P and cuts the plane of the circle at an angle $90° - \varphi$, where φ is a function at most of the two parameters of the circle. Moreover, for a congruence C the surface S is distinct from the envelope of the planes. When all the circles have the same radius and φ is constant, it is found that the planes of the circles are the osculating planes of a single parameter family of curves of constant torsion on S. The case where S degenerates into a curve is considered. For a cyclic system the curve of centers must be plane unless the circles are in the normal planes to the curve.

27. Professor John Eiesland: *A theorem in relativity.*

The following theorem is proved: Necessary and sufficient conditions that a centro-symmetric space with line element $-ds^2 = \varphi_2 dr^2 + \varphi_3(d\theta^2 + \sin^2\theta d\varphi^2) - \varphi_1 dt^2$, φ_1, φ_2 and φ_3 being arbitrary functions of r and t, shall be reducible to the static form are $\varphi_1\varphi_2 G_1^0 = (\partial\varphi_3/dr)(\partial\varphi_3/dt)$, and

$$\varphi_1\varphi_2\varphi_3(G_1 - G_0) = -\psi'\left[\varphi_1\left(\frac{d\varphi_3}{dr}\right)^2 + \varphi_3\left(\frac{d\varphi_3}{dt}\right)\right],$$

where ψ is an arbitrary function of φ_3. It follows that the Einstein solar field is necessarily static,* and that the same is true of other spaces of importance in relativity theory.

28. Professor John Eiesland: *On a generalization of Kummer's surface in odd n-space.*

Adopting as space element the ∞^n flats $\sigma_0 x_i - \sigma_{n/2} y_i + \rho_i = 0$, $\sigma_0 z + \sum \sigma_i y_i + \rho_{n/2} = 0$, $\sigma_{n/2} z + \sum \sigma_i x_i - \rho_0 = 0$ in

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* For an analytic proof of this proposition based on the existence-theorem for a pair of differential equations, see Birkhoff, *Relativity and Modern Physics*, Cambridge, 1923, pp. 253–256.

S_{n-1}, where $\sum \rho_i \sigma_i = 0$, the author considers the quadratic flat-complex $\varphi = \sum a_i(\rho_i^2 + \sigma_i^2) + 2\sum b_i \rho_i \sigma_i = 0$, and the singular surface of this complex. This surface is of the eighth order and eighth class, it is self dual, and has a 2-spread for locus of double-spreads. If Klein's coordinates $y_{2i+1} = \rho_i + \sigma_i$, $iy_{2n+2} = \rho_i - \sigma_i$ are introduced, the complex takes the form $\sum_1^{n+1} y_i^2 = 0, \sum_1^{n+1} k_i y_i^2 = 0$. The coordinates of the flats y_i may be represented as hyperelliptic functions of n parameters λ_i, and therefore also the coordinates of the singular surface may be similarly represented.

29. Mr. John McDonnell: *A note on chapter 2 of volume 3 of L. E. Dickson's History of the Theory of Numbers.*

In the above-mentioned chapter various theorems are stated concerning the expressibility of $4p$ in the form $a^2 + 27b^2$ when p is a prime $= 6n + 1$. In the present paper it is proved that p itself is expressible in the form $a^2 + 27b^2$ if $2^{(p-1)/3} \equiv 1 \pmod{p}$ and conversely.

30. Mr. A. D. Michal: *Integro-differential invariants of one-parameter groups of Volterra transformations.*

The author continues the discussion of integro-differential invariants, in particular functionals of the form $f[\overset{1}{\underset{0}{y}}, \overset{1}{\underset{0}{y'}}]$ which are "analytic," and shows that a necessary and sufficient condition that such a functional be invariant of a one-parameter group of linear functional transformations of Volterra type is that it satisfy a certain completely integrable equation in partial functional derivatives. The expansion of the solution of this equation is obtained.

31. Professor G. C. Evans: *The dynamics of monopoly.*

The author considers a problem in the dynamics of economics, namely, the change from one price to another under conditions of monopoly, when the demand function involves the rate of change of price as well as the price itself. The total profit over the stretch of time involved in the change shall be a maximum. The demand function is taken as of the first degree in the price and the rate of change of the price, and the cost is assumed to be a quadratic function of the amount produced.

R. G. D. Richardson,
Secretary.

AN INTRODUCTORY ACCOUNT OF THE ARITHMETICAL THEORY OF ALGEBRAIC NUMBERS AND ITS RECENT DEVELOPMENTS*

BY L. J. MORDELL

1. *Introduction.* In dealing with the subject of my lecture, I might have considered it from a purely logical point of view, and developed it in all its beauty in this manner. I think, however, it will be of more interest to you if I introduce it from the historical standpoint. Its beginnings date from Euler, who attempted to prove Fermat's statement, that the only integer solutions of the equation $y^2 + 2 = x^3$ were $x = 3$, $y = \pm 5$, by putting $x = a^2 + 2b^2$ and taking

$$y + \sqrt{-2} = (a + b\sqrt{-2})^3.$$

By equating irrational parts, he found

$$1 = b(3a^2 - 2b^2),$$

whence $b = 1$, $a = \pm 1$; but it is neither obvious nor true in general that all the integer solutions can be found in this way, —one used by Euler and Lagrange for some related questions. Then, about 1800, much interest was shown in the so-called law of quadratic reciprocity, first rigorously proved by Gauss; namely, that if p and q are two odd positive primes

$$\left(\frac{p}{q}\right)\left(\frac{q}{p}\right) = (-1)^{\frac{p-1}{2}\cdot\frac{q-1}{2}}.$$

The symbol (p/q) denotes $+1$ or -1, according as the congruence $x^2 \equiv p \pmod{q}$ is possible or impossible, and then p is called a quadratic or non-quadratic residue respectively of q. With certain extensions, this law is really equivalent to a reduction formula enabling us to calculate the value of the symbol (p/q), and forms the foundation of the theory of numbers.

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* Lecture read before the London Mathematical Society on January 18, 1923, and, by request of the program committee, before the American Mathematical Society, on September 6, 1923.

Obvious generalizations are suggested for other congruences, such as

$$x^3 \equiv p \pmod{q}, \qquad x^4 \equiv p \pmod{q}.$$

Taking the latter, it was found that the result appeared in a very complicated form, which, however, simplified remarkably if, instead of primes p and q, we considered their decompositions (when possible) in the form

$$p = a^2 + b^2,$$

or into complex factors $(a + ib)(a - ib)$. This led Gauss to the study of the arithmetic properties of complex numbers of the form $z = a + ib$, where a and b are integers. He proved that they did not differ essentially from those of ordinary integers, and showed that it was a very simple matter to define primes so that complex numbers could be factored uniquely, that if a product z_1z_2 were divisible by a complex prime z, then either z_1 or z_2 was divisible by z, etc.

Gauss also showed that if p was a prime of the form $4n + 1$, then the value of a (odd say) was given by the absolutely least residue satisfying the congruence

$$a \equiv (-1)^{n+1} \tfrac{1}{2} \frac{2n!}{(n!)^2} \qquad \pmod{p}.$$

This result was extended by Stern, Cauchy, Jacobi, and Eisenstein. The proofs depended upon complex numbers formed from other roots of unity than the fourth root i and applications were made to laws of reciprocity.

The most important facts concerning these numbers were discovered in connection with Fermat's last theorem on the insolubility in non-zero integers of the equation

$$x^p + y^p = z^p.$$

The left-hand side can be factored in the form

$$(x + y)(x + \zeta y)(x + \zeta^2 y) \cdots (x + \zeta^{p-1} y) = z^p,$$

where ζ is a complex pth root of unity; and it appeared to be a natural assumption, by analogy with elementary arithmetic, to put

$$x + \zeta y = (A + B\zeta + C\zeta^2 + \cdots)^p,$$

or perhaps to some multiple of the right-hand side, where $A, B, C, \cdots$ are ordinary integers. This involved the assumption that these algebraic numbers could be factored in a unique manner, an assumption which was, however, in general erroneous. This can be seen from a far more simple case:

$$\begin{aligned} 21 &= 3 \cdot 7 \\ &= (4 + \sqrt{-5})(4 - \sqrt{-5}) = (1 + 2\sqrt{-5})(1 - 2\sqrt{-5}), \end{aligned}$$

and it is easily verified that none of $3, 7, 4 \pm \sqrt{-5}, 1 \pm 2\sqrt{-5}$ can be split up into factors of the form $a + b\sqrt{-5}$ with a and b integers. Again

$$3^2 = (2 + \sqrt{-5})(2 - \sqrt{-5})$$

and $2 \pm \sqrt{-5}$ are not squares, and neither have a common factor of the form $a + b\sqrt{-5}$, nor can be split into factors of this form.

The primary object of the theory was to re-establish order in the chaos produced by the breakdown of the fundamental theorem upon which depends all the higher arithmetic. This was accomplished by Kummer in the special case of the algebraic numbers arising from the roots of unity, and more generally by several other writers. We shall give an account of Dedekind's method, since the main idea is not only easily explained, but is also very characteristic of mathematics, in generalizing a concept or a function by including it in a wider one. This idea is familar to all, e.g., $n!$ initially defined by $1 \cdot 2 \cdot 3 \cdots n$ when n is an integer, is generalized to

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$$\Gamma_{(n)} = \int_0^\infty e^{-x} x^{n-1} dx,$$

when n has its real part positive; and for all values of n by the well known infinite product. Again the chord of contact of the tangents from a point P to a conic can be generalized as the polar of P, giving from one point of view a simpler interpretation when the tangents are imaginary. Dedekind's idea was to consider groups of numbers which he called ideals, and with the obvious method of multiplying and dividing such groups as suggested by their definition, he showed that a unique factor law existed for his ideals, and that his results included as particular cases the fundamental laws of arithmetic and the correct arithmetical deductions to be drawn from relations involving algebraic numbers.

2. *Algebraic Numbers.* We must now define an algebraic number. The number θ is called an *algebraic number* if it is the root of an equation of the form

$$a\theta^n + b\theta^{n-1} + \cdots + l = 0,$$

where $a, b, \cdots, l$ are ordinary integers. If $a = 1$, θ is called an *algebraic integer,* and it is easy to show that this generalization is consistent, e.g., the sum, difference, product of integers are integers, and that if an algebraic integer is rational, it must be an ordinary integer. We may suppose that the equation in θ is irreducible in the field of rationality R defined by the ordinary integers, and that it has r_1 real roots and r_2 pairs of imaginary roots.

Any rational function of θ with rational coefficients, i.e., coefficients in R, can be reduced to the form

$$f = A_0 + A_1\theta + \cdots + A_{n-1}\theta^{n-1},$$

where $A_0, A_1, \cdots, A_{n-1}$ are rational numbers; and the assemblage of all such functions is referred to as the *field* or *Körper* $K(\theta)$. If f is an algebraic integer, the numbers conjugate to f are also algebraic integers, and by writing down the conjugate equations and solving, we find that $d(\theta)A_0$, $d(\theta)A_1, \cdots$ are rational integers where $d(\theta)$ is the discriminant of the equation in θ. Hence any algebraic integer in the field $K(\theta)$ can be written as

$$f = \frac{1}{d(\theta)}(a_0 + a_1\theta + \cdots + a_{n-1}\theta^{n-1}),$$

where $a_0, a_1, \cdots, a_{n-1}$ are integers. From this it follows that we can find n algebraic integers $\omega_1, \omega_2, \cdots, \omega_n$ called the base of the field, such that any algebraic integer f can be written in the form

$$f = x_1\omega_1 + x_2\omega_2 + \cdots + x_n\omega_n,$$

where $x_1, x_2, \cdots, x_n$ are rational integers. For example, in the field $K(\sqrt[3]{2})$, all integers are of the form $x_1 + x_2 \sqrt[3]{2} + x_3 \sqrt[3]{4}$, while in the field $K(\sqrt{5})$ they are of the form $x_1 + x_2(1 + \sqrt{5})/2$.

The base can be chosen in an infinite number of ways, but any one base can be derived from any other by a linear substitution in the ω's with integer coefficients and with a determinant unity. Hence the square of the determinant formed from the ω's and their conjugates, that is,

$$d = \begin{vmatrix} \omega_1^{(1)}, & \omega_2^{(1)}, & \cdots, & \omega_n^{(1)} \\ \omega_1^{(2)}, & \omega_2^{(2)}, & \cdots, & \omega_n^{(2)} \\ \cdot & \cdot & \cdot & \cdot \end{vmatrix}^2,$$

where $\omega_1^{(2)}$, etc., are the conjugates of $\omega_1^{(1)} = \omega_1$, etc., is an invariant of the bases, and is a rational integer called the discriminant d of the field. It is always greater than unity, and there are only a finite number of algebraic fields with a given discriminant d, as follows from an asymptotic formula given by Minkowski (as an example of his result stated further on), namely

$$d \sim \frac{1}{2\pi n}\left(\frac{\pi}{4}\right)^{2r_2} e^{2n - \frac{1}{6n}}.$$

3. *Units.* Among algebraic integers the most important are the units, i.e., the divisors of unity. For example, in the field $K(i)$, ± 1, $\pm i$ are divisors of unity, while in the field $K(\sqrt{2})$, $t + u\sqrt{2}$ is a unit if the integers t, u satisfy the equation $t^2 - 2u^2 = \pm 1$. It is well known that all the units are given by $\pm(1 + \sqrt{2})^n$, where n is any integer.

A similar theory holds in the general case, since it was proved by Dirichlet that any unit can be represented in the form $e_1^p e_2^q, \cdots$ for a finite number of values of $e_1, e_2, \cdots$, where $p, q, \cdots$ are any integers. For example, in the field $K(\sqrt[3]{2})$, the units are of the form $x + y\sqrt[3]{2} + z\sqrt[3]{4}$, where x, y, z are integers satisfying the equation

$$x^3 + 2y^3 + 4z^3 - 6xyz = \pm 1,$$

where the left-hand side is the norm of $x + y\sqrt[3]{2} + z\sqrt[3]{4}$, so that the theory gives the complete solution of this equation in x, y, z. It has lately been announced that it can be shown in this way that the equation $x^3 - ay^3 = 1$ never has more than one integer solution when a is given.

4. *Minkowski's Theorem.* It is Minkowski's theorem on linear forms, however, which is fundamental in the theory and has contributed greatly to its simplicity and elegance, namely, that integer values of x and y, not both zero, can be found so that

$$|ax + by| \leqq p, \qquad |cx + dy| < q,$$

where a, b, c, d are any real numbers, and $p, q > 0$, satisfy the equation

$$pq = \begin{vmatrix} a, & b \\ c, & d \end{vmatrix}.$$

There is of course the obvious extension to any number of variables $x, y, z, \cdots$.

No less than four distinct proofs have been given. The original proof by Minkowski is equivalent to the geometric theorem that any parallelogram in the x, y plane with one vertex at the origin and area $\geqq 1$ contains at least one lattice point on its sides or within its interior and is really a particular case of a far more general one. This proof, as well as the latest one just given by Siegel, which is analytic in character and depends upon trigonometric series, applies directly regardless of whether the coefficients are rational or not. The other proofs are arithmetic in character, first establishing the theorem for rational coefficients. Hilbert's proof depends upon Dirichlet's idea that if $n + 1$ objects are arranged in n groups, then one group will contain at least two objects. The proof by Hurwitz is a beautiful piece of arithmetic work, very characteristic of the author, showing that there are

$$\begin{vmatrix} a, & b \\ c, & d \end{vmatrix}$$

forms of the type $\lambda x + \mu y$, such that all forms $Ax + By$, with A, B integers, can be written in the form

$$\lambda x + \mu y + p(ax + by) + q(cx + dy),$$

where all coefficients, etc., are integers, and where a, b, c, d are given.

5. *Test for Algebraic Numbers.* I may say here a few words regarding the conditions that a given number θ should be an algebraic number of the nth degree, i.e., the root of an equation of the nth degree irreducible in R. Of the two kinds known, one states that a number θ cannot be an algebraic number of the nth degree if we can find an infinity of rational approximations p/q, such that

$$\left|\frac{p}{q} - \theta\right| < \frac{c}{q^\lambda},$$

where c is a given number, and where $\lambda > n$ according to Liouville, $\lambda \geqq n$ according to Thue in 1908, or finally $\lambda > 2\sqrt{n}$ according to Siegel in 1921. From this flow such results as that the equation $f(x, y) = c$ has only a finite number of integer solutions if f is an irreducible binary quantic in x, y of degree greater than 2. The proofs are very complicated, but very remarkable, depending only on elementary algebra.

The other types of results are due to Minkowski and Furtwängler and depend upon the investigation of the minima of the form

$$x_0 + x_1\theta + \cdots + x_{n-1}\theta^{n-1}$$

for integer values of $x_0, x_1, \cdots, x_{n-1}$, all numerically less than some number t. If now t takes the values $1, 2, \cdots$, we have a series of minima $m_1, m_2, m_3, \cdots$, which are such that the ratios $m_2/m_1, m_3/m_2, \cdots$ have only a finite number of values for all values of t.

6. *Ideals.* The algebraic integers in the field $K(\theta)$ form the foundation of all that follows, just as ordinary rational integers do in arithmetic, and the word *integer* hereafter refers to the integers in the field $K(\theta)$.

Let $\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_p$ be any given integers; then the *ideal* A is the group of integers defined by $\lambda_1\alpha_1 + \lambda_2\alpha_2 + \cdots$, where $\lambda_1, \lambda_2, \lambda_3, \cdots$ are any integers, a fact expressed by the notation

$$A = [\alpha_1, \alpha_2, \alpha_3, \cdots, \alpha_p].$$

The ideals A and $B = [\beta_1, \beta_2, \beta_3, \cdots, \beta_q]$ are equal, written $A = B$ if every number of A is included in B and conversely; i.e., if integers λ, μ can be found so that

$$\beta_1 = \lambda_{11}\alpha_1 + \lambda_{12}\alpha_2 + \cdots, \qquad \beta_2 = \lambda_{21}\alpha_1 + \lambda_{22}\alpha_2 + \cdots,$$

and

$$\alpha_1 = \mu_{11}\beta_1 + \mu_{12}\beta_2 + \cdots, \qquad \alpha_2 = \mu_{21}\beta_1 + \mu_{22}\beta_2 + \cdots.$$

An ideal is called a *principal ideal* if it can be written in the form $[\alpha]$, so that it consists of all the integers divisible by α. Further the principal ideals $[\alpha]$ and $[\beta]$ are equal if and only if α and β are associated integers, i.e., $\alpha = \beta e$, where e is a unit.

In particular, if $\alpha = 1$, we have as the unit ideal [1] all the integers in $K(\theta)$.

We can now extend many arithmetic concepts to ideals. Thus the product AB of the ideals A and B is defined as the ideal C formed from the numbers obtained by multiplying every number of A by every number of B; and we write $C = AB$. The commutative law is obviously satisfied, so that we can write $A \times A = A^2$, $A \times A \times A = A^3$, etc., while A^0 stands for the unit ideal [1].

Division is defined as the inverse of multiplication, so that the ideal C is divisible by the ideal A if an ideal B can be found so that $C = AB$. An ideal P is called a *prime ideal* if it is divisible by only itself and [1], the unit ideal. Finally, two ideals are called *prime to each other* if they have no common divisor except [1].

The fundamental theorem in the theory of ideals states that an ideal can be factored in only one way, apart of course from the order of the factors. Many important consequences follow just as in elementary number theory. The proof can be presented in several different ways, requiring in any case a long chain of subsidiary propositions. In Hurwitz' method the important steps are as follows:

(1) Corresponding to any ideal A we can find an ideal B so that AB is a principal ideal $[\alpha]$ where α is a positive rational integer.

(2) From the equality $[\gamma]A = [\gamma]B$, we have $A = B$, and hence from $CA = CB$, also $A = B$. From these we show that if C is divisible by A, every number of C is included in A, and conversely.

It is then shown that an ideal A has only a finite number of divisors, since a given rational integer α is a member of only a finite number of ideals.

The next step is to show that if an ideal P is a divisor of AB, then either A or B is divisible by P. This depends upon the fact that the greatest common divisor of the ideals A, B is given by $[\alpha_1, \alpha_2, \cdots, \alpha_p, \beta_1, \beta_2, \cdots, \beta_q]$. The result follows immediately.

An ideal can be factored by a definite and direct process. We can then factor any algebraic integer I by factoring the principal ideal $[I]$. Suppose we find

$$[I] = A^\lambda B^\mu C^\nu \cdots,$$

where $A, B, C, \cdots$ are different prime ideals. Then if $A, B, C, \cdots$ are principal ideals $[\alpha], [\beta], [\gamma], \cdots$, we have the result that

$$I = \epsilon \alpha^\lambda \beta^\mu \gamma^\nu \cdots,$$

where ϵ is a unit. If, however, one at least of the ideals $A, B, C, \cdots$ is not a principal ideal, the integer I cannot be factored in the ordinary sense, though the ideal $[I]$ can be. It is for this reason that I cannot be considered as a prime, and that a unique factorization law that would naturally suggest itself does not hold in the theory of algebraic numbers.

7. *Congruences and Norms.* We can now consider congruences with respect to ideals. A number μ is said to be divisible by the ideal A if μ is one of the numbers forming the ideal A, or, what amounts to the same thing, if the principal ideal $[\mu]$ is divisible by A; and we write

$$\mu \equiv 0 \pmod{A}.$$

Hence we mean by the congruence

$$\mu \equiv \nu \pmod{A}$$

that $[\mu - \nu]$ is divisible by A, or that $\mu - \nu$ belongs to the ideal A. The ideal A is only a part of the integers of $K(\theta)$, and the number of incongruent integers (mod A) is finite and is denoted by $N(A)$. In particular, if A is a principal ideal $[\mu]$, $N(A)$ is the absolute value of the product of μ by its conjugates. For example, in the field $K(i)$, if ω is a prime of the form $a + ib$, where $p = a^2 + b^2$ is a rational prime $\equiv 1 \pmod 4$ and where a, b are rational integers, the number of incongruent integers (mod ω) is

$$(a + ib)(a - ib) = p.$$

If μ is a rational integer, the number $N(\mu)$ becomes $|\mu^n|$.

The norm satisfies the law

$$N(A)N(B) = N(AB).$$

To find a simple formula for the norm of an ideal, we first prove that an ideal has a base, i.e., n integers $\alpha_1, \alpha_2, \cdots, \alpha_n$ can be found so that all the numbers of A can be written in the form

$$\lambda_1\alpha_1 + \lambda_2\alpha_2 + \cdots + \lambda_n\alpha_n,$$

where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are rational integers, and further that the determinant

$$\begin{vmatrix} \alpha_1^{(1)}, \alpha_2^{(1)}, \cdots, \alpha_n^{(1)} \\ \alpha_1^{(2)}, \alpha_2^{(2)}, \cdots, \alpha_n^{(2)} \\ \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{vmatrix}$$

formed from the base and its conjugates is an invariant of the ideal. The numbers $\alpha_1, \alpha_2, \cdots, \alpha_n$ of the base can be expressed in terms of the base $\omega_1, \omega_2, \cdots, \omega_n$ of the field by means of the equations

$$\begin{aligned} \alpha_1 &= c_{11}\omega_1 + c_{12}\omega_2 + \cdots, \\ \alpha_2 &= c_{21}\omega_1 + c_{22}\omega_2 + \cdots, \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned}$$

where the c's are rational integers. The determinant $|c_{kl}|$ is the norm of the ideal A. We can also write

$$|\alpha_{kl}|^2 = dN^2(A),$$

a most useful equation.

The norm of a prime ideal P takes a very simple form. For P must be a divisor of a rational prime p which is the smallest rational integer divisible by P. Hence we have

$$N(P) = p^f, \quad \text{where} \quad 1 \leqq f \leqq n.$$

The integer f is called the *grade* of the ideal. It should be noted that only a finite number of ideals have a given norm.

There are of course an infinite number of ideals, but they can be divided into a finite number of classes. Thus an ideal A will be called equivalent to B, i.e., they are in the same class, if integers α, β can be found so that

$$[\alpha]A = [\beta]B,$$

a fact expressed by writing $A \sim B$.

All the principal ideals are equivalent, and they constitute the principal class. Further, there are only a finite number of classes, as follows from the fact that every class contains an ideal whose norm $\leqq \sqrt{d}$. This is a simple deduction from the fact, depending upon Minkowski's theorem, that every ideal A contains a number a such that

$$|N(a)| \leqq N(A)\sqrt{d}.$$

We could also in this way actually calculate H, the number of ideal classes. The classes of ideals form an ordinary abelian group.

The whole theory can be developed by proving that H is finite, and then deducing the unique factorization law.

8. *Application to Indeterminate Equations.* This number H, and the fact that it is finite, are of the greatest importance in the applications to indeterminate equations. For if A is any ideal, then A^H belongs to the principal class, i.e., $A^H \sim [1]$. Conversely, if $A^n \sim [1]$, and n is prime to H, it follows that A is a principal ideal. Hence, if we wish to draw any conclusion from the equality $ab = c^n$ in algebraic integers, we must first write it as an equation in ideals $[a][b] = [c]^n$. Therefore, if the ideals $[a]$, $[b]$ have a common factor, say a principal ideal $[d]$, we must have, say,

$$[a] = [d]A^n, \qquad [b] = [d]^{n-1}B^n,$$

where $[c] = AB$; and the ideals A and B will be principal ideals only if n is prime to H. We then have

$$A = [x], \qquad B = [y],$$
$$[a] = [d][x]^n, \qquad [b] = [d]^{n-1}[y]^n,$$

whence $a = dx^n e$, where e is some unit.

For example, in Fermat's equation in rational integers,

$$x^p + y^p = z^p,$$

or

$$(x + y)(x + \zeta y)(x + \zeta^2 y) \cdots = z^p,$$

where $\zeta = e^{2\pi i/p}$, we consider first the case when x, y, z are all prime to p, that is, the greatest common factor of the ideals $[x + \zeta y], [x + \zeta^2 y], \cdots$, is unity. Hence, if p is prime to the number of ideal classes in the field $K(\zeta)$, we have

$$x + \zeta y = ea^p,$$

where e is a unit and a is an algebraic integer. It is not very difficult to prove from this equation that Fermat's equation is impossible. Similarly, when one of x, y, z is divisible by p, all, of course, on the assumption that p is prime to H.

Another illustration is given by the classical indeterminate equation

$$az^2 = x^4 + bx^3y + cx^2y^2 + dxy^3 + ey^4,$$

where a, b, c, d, e are given rational integers and x, y, z are unknown rational integers. If θ is a root of the equation

$$\theta^4 + b\theta^3 + c\theta^2 + d\theta + e = 0,$$

az^2 has a factor $x - \theta y$, so that we have the equation in ideals

$$[x - \theta y] = \mu\tau^2,$$

where μ is one of a finite number of ideals and τ is an unknown ideal. Since the number of ideal classes is finite we can put $\tau = \alpha v/\beta$, where α, β are integers, and v is one of a finite number of ideals. Then μv^2 must be one of a finite number of principal ideals, say $[\gamma]$, so that

$$[x - \theta y] = \alpha^2[\gamma]/\beta^2,$$

whence

$$x - \theta y = \epsilon\alpha^2\gamma/\beta^2,$$

where e is a unit. Since all units can be expressed in the form $\epsilon_1 \epsilon_2{}^2$ for a finite number of values of ϵ_1, we deduce an equation of the form

$$x - \theta y = \nu\alpha^2,$$

where ν is one of a finite number of integers and α is an unknown integer. If this has an infinite number of solutions, we have for a particular solution

$$x_0 - \theta y_0 = \nu\alpha_0^2,$$

whence, by multiplication, we can deduce an equation of the form

$$(x - \theta y)(x_0 - \theta y_0) = M^2(a + b\theta + c\theta^2 + d\theta^3),$$

where x, y, a, b, c, d are unknown rational integers and x_0, y_0, M are given rational integers. It was from an equation of this form (e.g., with $x_0 = 1, y_0 = 0$) that I showed that the method of infinite descent applied to the original equation, and hence to the homogeneous ternary cubic

$$f(x, y, z) = 0,$$

i.e., that all its rational solutions could be derived from a finite number by the classic method.*

9. *The Class Number.* The problem of finding the number H of ideal classes is a very interesting and difficult one. Analytically H is a multiple of the residue at $s = 1$ of the function defined when the real part of s is greater than one, by

$$f(s) = \sum \frac{1}{N[A]^s},$$

where the summation refers to all the ideals of the field $K(\theta)$. This is deduced from the series

$$f_L(s) = \sum_A \frac{1}{N(A)^s},$$

the summation referring to all the ideals in the class L, and the residue at $s = 1$ being independent of the class L. We may also write

$$f_L(s) = N(B)^s \sum_\mu \frac{1}{N(\mu)^s},$$

where the summation refers to all non-associated integers μ divisible by the ideal B, and B is an ideal in the class L^{-1}.

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* The same method shows that $Ey^2 = Ax^3 + Bx^2 + Cx + D$ has only a finite number of integer solutions if the right side has no squared factor in x.

The function $f(s)$ is the analog of the ordinary Riemann zeta function. Its chief properties remained unknown for many years, and their investigation was one of the problems proposed by Hilbert in his address in 1900 to the International Mathematical Congress at Paris.

In 1916, it was shown by Hecke that $f(s)$ represented a function of s which can be continued throughout the s plane, whose only singularity is a simple pole at $s = 1$, and which also satisfies a very simple functional relation. His method can be illustrated by considering the ordinary ζ function

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots.$$

By using the gamma function, we can express this as a definite integral

$$\int_0^\infty \xi^{\frac{s}{2}-1}\theta(\xi)d\xi,$$

where $\theta(\xi)$ is practically a theta function. The range of integration is split into $\int_1^\xi + \int_0^1$. In the former, ξ is changed into $1/\xi$ and the classical transformation formula for $\theta(1/\xi)$ is applied. The result at once follows. Moreover, a simple functional equation between $\zeta(s)$ and $\zeta(1-s)$ is apparent.

The same method applies to $f_L(s)$ by noting that

$$N(\mu) = \mu_1\mu_2 \cdots \mu_n,$$

so that $f_L(s)$ can be transformed into a multiple integral with limits ∞, 0 by writing

$$\Gamma_{(s)}k^{-s} = \int_0^\infty e^{-kt}t^{s-1}dt$$

if k is real and positive, $R(s) > 0$. The great difficulty was to express the fact that the summation refers to the non-associated integer μs, i.e., that only one number of the group $e_1{}^pe_2{}^q \cdots \mu$ arising from any integer values of p, q appears in the summation. By writing the limits of integration as

$$\int_0^1 + \int_1^2 + \int_2^3 + \cdots,$$

Hecke transformed the integral into a theta series with n variables associated with the ideal, where the summation now refers to all the integers μ (not merely the non-associated one). Moreover, since the multiple theta series also had a simple transformation formula, he was able to find a simple functional relation between $f(s)$ and $f(1-s)$, to show that $f(s)$ exists all over the s plane, and has a simple pole at $s = 1$. Siegel has lately shown that these results can be found in another way, wherein the units make no appearance. The same method applies to many functions associated with or derived from the Dedekind zeta function, such as the series

$$\sum_A \frac{\chi(A)}{[N(A)]^s},$$

where now $\chi(A)$ is a root of unity associated with the ideal A, and many results in the ordinary theory of prime numbers can be extended to the case of prime ideals. For example, the prime number theorem, which states that the number of primes less than x is asymptotically equal to $x/\log x$, is equivalent to the fact that the Riemann zeta function has only a simple pole at $s = 1$ and no zeros in a contour whose right-hand boundary is say $z = 2$, and whose left-hand boundary approaches $z = 1$ from the left according to the law

$$y = 1 - \frac{a}{\log x},$$

where a is a constant. The Dedekind zeta function has practically the same properties as the Riemann function whence results the same asymptotic formula for ideals as for ordinary primes, e.g., the number of prime ideals whose norm $\leqq x$ is asymptotically equal to $x/\log x$. Further Hecke was able to prove results such as that the indefinite form $ax^2 + bxy + cy^2$ represents an infinite number of primes in any given sector of the xy plane whose center is at 0 and whose radius is infinite. The θ functions dealt with by Hecke are associated with ideals and algebraic numbers in a very simple and elegant manner. A simple case is that in which we have

$$\theta(t,A) = \sum_{\mu\equiv 0 \pmod{A}} e^{-c|\mu_1|^2 t_1 - c|\mu_2|^2 t_2 - \cdots - c|\mu_n|^2 t_n},$$

where the summation refers to all the numbers $\mu = \mu_1$ of an ideal A, and $\mu_2, \mu_3, \cdots$ are the conjugates of μ, and c is a constant depending on the ideal A. Then we may write

$$\theta(t, A) = \frac{1}{\sqrt{t_1 t_2 \cdots t_n}} \theta\left(\frac{1}{t}, B\right),$$

where B is an ideal derived very simply from A, namely $AB = 1/D$, where D is the *grund* ideal, really an ideal whose norm is the discriminant d.

These θ functions led Hecke to the consideration of Gauss' sums in any algebraic field, for which they are as important as are the ordinary Gauss' sums in the elementary theory of numbers. He showed that the reciprocity formula for them follows from the functional equation for the zeta function, and that the law of quadratic reciprocity for any algebraic field is a simple consequence of his general methods. His functional equations have enabled him to prove a number of striking results, both arithmetic and analytic. One of the latter is that if $R(x)$ is the fractional part of x so that $0 \leqq R(x) < 1$, while $x - R(x)$ is an ordinary integer, then the Dirichlet series

$$\sum_{m=1}^{\infty} \frac{R(ma) - 1/2}{m^s},$$

where $a = \sqrt{d}$ or $1/\sqrt{d}$, represents a meromorphic function of s, analytic for $R(s) > 0$, while if $R(s) \leqq 0$, it has simple poles at the points represented by the formula

$$s = -2k \pm \frac{2n\pi i}{\log \eta}, \quad (n, k = 0, 1, 2, \cdots),$$

where η is the fundamental unit or its square in the field $K(\sqrt{d})$.

10. *Representations as Sums of Squares.* Siegel has shown that the methods introduced by Hardy and Littlewood into the analytic theory of numbers can also be extended to similar questions involving algebraic numbers. Thus the question of finding approximate formulas for the number of representations of a given rational integer n as a sum of say 5 squares is equivalent to finding the coefficient of x^n in the expansion

of $f(x) = (1 + 2x + 2x^4 + \cdots)^5$. This is given by a contour integral around a circle whose radius is taken to be very nearly one, say $1 - 1/n$. The range of integration 2π and 0, or say 1, 0, is split up into arcs according to the Farey method of division of order, say $[\sqrt{n}]$. If the fraction p/q is associated with one of these arcs, the integrand is evaluated approximately at $x = e^{2\pi i p/q}$ and the sum of the resulting integrals is proved to be a genuine approximation. Siegel has shown that if we seek the number of representations of an algebraic integer as a sum of squares of algebraic integers (when the field and its conjugates are real), the θ function $1 + 2x + 2x^4 + \cdots$ can be replaced by the θ series considered by Hecke. For example, in a quadratic field, the coefficient involves a double integral over the unit square. This square can be subdivided in a method similar to the ordinary Farey method into a number of small regions, and in each of these regions an approximate value is taken for the integrand, and the resulting integral again gives a genuine approximation.

11. *Laws of Reciprocity.* Finally the general laws of reciprocity in any field, i.e., the investigation of the congruence

$$x^n \equiv p \pmod{Q},$$

where Q is a given ideal, p a given algebraic integer, x an unknown algebraic integer, and n a given rational integer, are some of the most successful, abstruse, and far reaching results of the ideal theory, giving one a glimpse of regions so remote that apparently many years will elapse before our efforts will bring us within measurable distance.

First consider the congruence $x^2 \equiv q \pmod{p}$ in rational numbers, where p and q are odd primes. As factorization of $x^2 - q$ suggests $\sqrt{q}$, let us examine the meaning of this congruence in the field $K(\sqrt{q})$. If it is possible, it is equivalent to saying that the prime p factors in the field $K(\sqrt{q})$. By associating with any ideal a certain number of roots of unity, say $e_1, e_2, \cdots, e_k$, we can divide the ideal classes into genera; and conversely, if these units are given and satisfy an equation of consistency $e_1 e_2 \cdots e_k = 1$, we can find a class of ideals associated with the given units. Not only can we prove the law of quadratic reciprocity in this way, as was done by Kummer, but the ideas involved are so general that they have been extended by Hilbert and Furtwängler to the law of reciprocity in any field. Many new ideas, however, are involved in the proof, e.g., if $x^2 \equiv q \pmod{p}$ where now x, q are integers, and p is an ideal in the field $K(\theta)$, we have to investigate the properties of the algebraic field $K(\sqrt{q})$. It is of course obvious that $\sqrt{q}$ satisfies an equation of degree $2n$, but it is more convenient to consider it as a number satisfying a quadratic equation, the coefficients being integers in $K(\theta)$, so that $\sqrt{q}$ generates a quadratic field relative to the field $K(\theta)$. So it is more convenient to consider the ideals in the field $K(\sqrt{q})$ as quadratic ideals relative to the field $K(\theta)$. If the field $K(\theta)$ satisfies certain very special conditions, the study of the relative field $\sqrt{q}$ leads to the law of quadratic reciprocity in the field. All the laws of reciprocity, quadratic, cubic, etc., can be deduced when the proper relative field is known. This is a question of great difficulty and importance.

Great progress has been made arithmetically as far as the laws of reciprocity are concerned. In particular, the relative discriminant of these fields is unity and the relative Galois group is isomorphic with the group of the ideal classes in $K(\theta)$. Analytically, however, it has only been done in a few cases. The problem is equivalent to questions such as the following. Take the equation $x^p = 1$ where p is a prime. This is an abelian equation, i.e., all its roots are rationally represented in terms of one of them, e.g., $\zeta, \zeta^2, \zeta^3, \zeta^4, \cdots$, or, say, $f_1(\zeta)$, $f_2(\zeta), f_3(\zeta), \cdots$, and it is obvious that

$$f_a[f_b(\zeta)] = f_b[f_a(\zeta)].$$

Then there is a theorem which states that the root of any abelian equation whose coefficients are rational integers, can be expressed rationally with ordinary rational coefficients in terms of roots of unity. The next stage is that the roots of any abelian equation whose coefficients are imaginary quadratic integers in the field $K(\omega)$ can be expressed rationally in terms of $j(\omega)$, where $j(\omega)$ is the well known modular function, that is, by means of the complex multiplication of elliptic functions.

Beyond this, however, we at present cannot go. But it is obvious what a field of research is suggested for the future.

Some of the relative fields are given by the equations in elliptic functions dealing with the subdivision of the periods, and also by the modular functions of several variables, an idea due to Hilbert, and developed by Blumenthal and Hecke and intimately connected with the θ function in the Riemann theory of algebraic functions.

The final law of reciprocity can be stated in all its generality in a remarkably simple form. For example, the law of quadratic reciprocity in any field $K(\theta)$ is equivalent to the theorem that the equation

$$ax^2 + by^2 + cz^2 = 0,$$

where a, b, c are given and x, y, z are unknown coprime integers in the field, is possible if and only if the congruence

$$ax^2 + by^2 + cz^2 \equiv 0 \pmod{P}$$

can be satisfied if P is any ideal in the field. A simple proof of this would of course lead to an easy arithmetical proof of the laws of reciprocity, and it is well worth the attention of mathematicians.

Finally, we may state that Siegel has made recently an interesting application of the law by showing that every algebraic integer can be expressed as the sum of 4 squares of algebraic numbers provided it is totally positive, that is, those of the conjugates that are real must be positive. Waring's theorem also has been extended to algebraic numbers.*

THE UNIVERSITY OF MANCHESTER, ENGLAND

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* For references on this and other topics mentioned above, see the *Report on Algebraic Numbers*, BULLETIN OF THE NATIONAL RESEARCH COUNCIL, February, 1923; H. Bohr and H. Cramer, *Die neuere Entwickelung der analytischen Zahlentheorie*, ENCYKLOPÄDIE, II C 8; Hecke, *Vorlesungen über die Theorie der Algebraischen Zahlen*; Mordell, *Indeterminate equations of the third degree*, SCIENCE PROGRESS, July, 1923.

INTEGRAL SOLUTIONS OF $x^2 - my^2 = zw$*

BY L. E. DICKSON

1. *Statement of the Theorem.* In this BULLETIN (vol. 27 (1920–1), p. 361) I stated that all integral solutions of the equation

$$x^2 - my^2 = zw$$

are obtained by multiplying the right members of

$$(1)\quad z = el^2 + 2flq + gq^2, \qquad w = en^2 - 2fnr + gr^2,$$

$$(2)\quad x = \pm (eln + fnq - flr - gqr), \qquad y = lr + nq,$$

by the same arbitrary integer, where e, f, g take only those sets of integral values (finite in number) for which the first form (1) is a reduced quadratic form having the same discriminant $4m$ as $x^2 - my^2$, so that

$$(3)\qquad f^2 - eg = m.$$

In other words, we employ a single form $el^2 + \cdots$ from each class of quadratic forms of discriminant $4m$. The number of such classes is therefore the number of sets of formulas (1), (2) giving all integral solutions of $x^2 - my^2 = zw$. If we permit the interchange of z and w, we need retain only a single sign in (2). For, if we replace l, q, n, r by $n, -r, -l, q$, respectively, we find that z and w are interchanged, y is unaltered, and x is replaced by $-x$.

In the paper cited, I was led to the above theorem by the theory of ideals, and I gave a proof when there is a single class of quadratic forms.† I stated that a simpler proof of the general theorem follows by composition. I have since found the following still simpler proof.

2. *A Simplification.* Since we may lay aside a common factor of x, y, z, w, consider $X^2 - mY^2 = ZW$, where X, Y, Z, W

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* Presented to the Society, December 1, 1923.

† A complete proof of the general theorem by means of ideals has been found recently by G. E. Wahlin.

are integers without a common factor greater than 1. We may write

$$X = dx, \qquad Y = dy, \qquad d = \delta D, \qquad Z = \delta z_1,$$

where x and y are relatively prime, and likewise D and z_1, while δ is prime to W. Then $d^2(x^2 - my^2) = \delta z_1 W$, so that $z_1 W$ is divisible by δD^2. Hence $z_1 = \delta z$, $W = D^2 w$. We have been led back to our initial equation $x^2 - my^2 = zw$ with the simplification that x and y are now relatively prime.

3. *Argument.* If a prime divided both y and w, it would divide x^2 and hence also x, contrary to the last result. Since $y = l$ and $w = g$ are therefore relatively prime, there exist integers f and q such that $x = fl + qg$. Inserting these values of x, y and w in $x^2 - my^2 = zw$, we get

$$(4) \qquad zg = (f^2 - m)l^2 + 2fglq + g^2q^2.$$

Since g is prime to l, we have (3), where e is an integer. Cancellation of g gives (1_1).

Let the substitution, with integral coefficients,

$$l = rl_1 + nq_1, \qquad q = sl_1 + tq_1, \qquad rt - ns = 1,$$

replace the first form (1) by a reduced form

$$F(l_1, q_1) = e_1l_1^2 + 2f_1l_1q_1 + g_1q_1^2.$$

The latter is therefore replaced by the former by the inverse substitution

$$l_1 = tl - nq, \qquad q_1 = -sl + rq.$$

Taking $l = 0$ for the moment, we see that the coefficient $g = w$ of q^2 in (1_1) is

$$F(-n, r) = e_1n^2 - 2f_1nr + g_1r^2.$$

Moreover, we have $y = l = rl_1 + nq_1$. Dropping the subscripts, we have the expressions for z, w, y in (1), (2). While the expression (2) for x may be derived from $x^2 - my^2 = zw$ by direct computation, it may be obtained more simply as follows. We have

$$(5) \qquad gz = (fl + qg)^2 - ml^2, \qquad gw = (gr - fn)^2 - mn^2,$$

the first being another form of (4). The product of their linear factors is

$$(6)\qquad (fl + qg + l\sqrt{m})(gr - fn + n\sqrt{m}) = -g(\pm x - y\sqrt{m}),$$

in view of (2). Taking the product of (6) by the identity derived from it by changing the sign before $\sqrt{m}$, and removing the factor g^2, we get $zw = x^2 - my^2$. Hence x has the values in (2). This proves that conversely (1) and (2) are solutions for all values of l, q, n, r.

4. *Conclusion.* It follows that every set of integral solutions of $x^2 - my^2 = zw$ (with x and y relatively prime) is obtained by multiplying the second members of (1), (2) by the same arbitrary integer. The restriction in parenthesis may be discarded. For, by §2, the general solution of $X^2 - mY^2 = ZW$ in integers is given by

$$X = \delta Dx,\qquad Y = \delta Dy,\qquad Z = \delta^2 z,\qquad W = D^2 w.$$

Let us write

$$L = \delta l,\qquad Q = \delta q,\qquad N = Dn,\qquad R = Dr.$$

Then the values of Z, W, X, Y are derived from (1), (2) by replacing l, q, n, r by L, Q, N, R, respectively.

5. *A Companion Theorem.* The above proof applies without essential change to the solutions of the equation

$$(7)\qquad x^2 + xy + \tfrac{1}{4}(1 - m)y^2 = zw,\qquad m \equiv 1 \pmod 4,$$

and hence leads to the companion theorem of the former paper.

6. *Generalization.* As a generalization, consider

$$(8)\qquad ax^2 + 2bxy + cy^2 = zw.$$

Multiply by a and write $m = b^2 - ac$, $\xi = ax + by$. Then

$$(9)\qquad \xi^2 - my^2 = z\cdot aw.$$

Hence z, aw, ξ, y are given by the products of the right members of (1), (2) by the same arbitrary integer. The second form (1) will be divisible by a prime factor p of a if and only if n and r satisfy two linear homogeneous congruences modulo p (which may coincide); they serve to express n and r in one or two ways as linear homogeneous functions of new parameters

N and R. The initial form (1_2) becomes one or two new quadratic forms in N and R. We proceed similarly with a prime factor of a/p, etc. Finally, we obtain formulas for x from $ax = \xi - by$. We conclude that all integral solutions of (8) are products of the same arbitrary integer by the numbers obtained from a finite number of sets of four expressions each quadratic in four arbitrary parameters. The explicit formulas will be discussed on another occasion.

THE UNIVERSITY OF CHICAGO

ON THE REALITY OF THE ZEROS OF A λ-DETERMINANT *

BY R. G. D. RICHARDSON

Some of the best-known theorems of algebra are centered around the zeros of the polynomial in λ,

$$(1) \qquad \begin{vmatrix} a_{11} - \lambda & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} - \lambda & \cdots & a_{2n} \\ \cdot & \cdot & \cdot & \cdot \\ a_{n1} & a_{n2} & \cdots & a_{nn} - \lambda \end{vmatrix} .$$

In the classical case of the determinant connected with the equations of secular variations, where the elements a_{ij} are real and the determinant $|a_{ij}|$ formed from (1) by omitting the λ's is symmetric $(a_{ij} = a_{ji})$, these zeros turn out to be real. This theorem concerning the reality of the zeros has been extended † to the case where a_{ij} and a_{ji} are conjugate complex $(a_{ij} = \bar{a}_{ji})$. It is proposed in this note to extend it to a still more general case which has arisen in some investigations concerning pairs of bilinear forms just completed by the author. This generalization consists in allowing the coefficients of the λ's to be n^2 in number instead of n as in (1), of allowing them to be various and complex instead of all unity, and of bordering the determinant by m rows and m columns. The

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* Presented to the Society October 27, 1923.

† Cf. Kowalewski, *Einführung in die Determinantentheorie*, p. 130.

method of proof is extremely elegant and it is worthy of notice that for the special case of the known theorems concerning (1), the proof itself is much simpler than any previously given.

Let us consider the determinant*

$$(2)\qquad \begin{vmatrix} a_{11}-\lambda b_{11} & \cdots & a_{1n}-\lambda b_{1n} & a_{1,\,n+1} & \cdots & a_{1,\,n+m} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n1}-\lambda b_{n1} & \cdots & a_{nn}-\lambda b_{nn} & a_{n,\,n+1} & \cdots & a_{n,\,n+m} \\ a_{n+1,\,1} & \cdots & a_{n+1,\,n} & 0 & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ a_{n+m,\,1} & \cdots & a_{n+m,\,n} & 0 & \cdots & 0 \end{vmatrix},$$

under the following hypotheses:

(*a*) the elements a_{ij} are conjugate complex;

(*b*) the elements b_{ij} are conjugate complex and the corresponding hermitian form

$$\sum\nolimits_{ij}^{1,\,n} b_{ij} z_i \bar{z}_j$$

is definite.†

Under these hypotheses it is proposed to show that the zeros of the $(n-m)$th degree polynomial (2) in λ are real.

PROOF. Consider the $n+m$ homogeneous equations

$$(3)\qquad \begin{cases} \displaystyle\sum_{j}^{1,\,n}(a_{ij}-\lambda b_{ij})x_j + \sum_{k}^{n+1,\,n+m} a_{ik}x_k = 0 & (i=1,\cdots,n), \\ \displaystyle\sum_{k}^{1,\,n} a_{ik}x_k = 0 & (i=n+1,\cdots,n+m), \end{cases}$$

in $n+m$ variables x, whose coefficients are the elements of the determinant (2). The necessary and sufficient condition that

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* It is obvious that m must be less than n; otherwise the polynomial would be identically zero, and any theorem concerning the roots necessarily trivial.

† A necessary and sufficient condition that it be positive definite is that the terms of the sequence of $n+1$ determinants

$$1,\quad b_{11},\quad \begin{vmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{vmatrix},\quad \begin{vmatrix} b_{11} & \cdots & b_{13} \\ \cdot & \cdot & \cdot \\ b_{31} & \cdots & b_{33} \end{vmatrix},\quad \cdots,\quad \begin{vmatrix} b_{11} & \cdots & b_{1n} \\ \cdot & \cdot & \cdot \\ b_{n1} & \cdots & b_{nn} \end{vmatrix}$$

all have the same sign; the condition that it be negative definite is that these terms alternate in sign.

(3) have a solution is that (2) vanish. Hence for any zero of (2), the equations (3) have a solution $x_1, x_2, \cdots, x_{n+m}$, not identically zero. Multiplying the ith equation (3) by $\bar{x}_i$ $(i = 1, \cdots, n+m)$, and adding, we find a relation which may be written in the form

$$\begin{aligned}\sum_{ij}^{1,n} a_{ij}x_j\bar{x}_i - \lambda \sum_{ij}^{1,n} b_{ij}x_j\bar{x}_i \qquad\qquad & \\ + \sum_{i}^{1,n} a_{ik} \sum_{k}^{n+1,n+m} x_k\bar{x}_i + \sum_{i}^{n+1,n+m} \sum_{k}^{1,n} a_{ik}x_k\bar{x}_i = 0. & \end{aligned} \tag{4}$$

Now, by hypothesis (*a*), the terms $a_{ik}x_k\bar{x}_i$, $a_{ki}\bar{x}_kx_i$ are conjugate and their sum is real. Hence the first and second sums in relation (4) are hermitian forms and necessarily real, each of the n^2 terms in each sum being matched by its conjugate in the same sum. Further, since

$$\sum_{i}^{n+1,n+m} \sum_{k}^{1,n} a_{ik}x_k\bar{x}_i \equiv \sum_{k}^{n+1,n+m} \sum_{i}^{1,n} a_{ki}\bar{x}_kx_i,$$

it follows that each of the nm terms in the third sum is matched with a conjugate among the nm terms in the fourth and hence that the sum of these two sums is real. Finally by hypothesis (*b*), the second sum is different from zero (being zero only when all the x's vanish). Hence λ is real.

THEOREM. *The zeros of the* λ-*determinant* (2) *are real provided hypotheses* (*a*) *and* (*b*) *are satisfied.*

In the special case of the determinant of secular variations, (3) and (4) respectively take the forms

$$\sum_{j}^{1,n} a_{ij}x_j = \lambda x_i, \quad (i = 1, \cdots, n); \qquad \sum_{ij}^{1,n} a_{ij}x_ix_j = \lambda \sum_{i}^{1,n} x_i^2.$$

The proof of the reality of the λ's is thus notably simple.

BROWN UNIVERSITY

BOUTROUX ON MATHEMATICAL IDEALS

L'Idéal Scientifique des Mathématiciens dans l'Antiquité et dans les Temps Modernes. By Pierre Boutroux. Paris, Felix Alcan, 1920. 274 pp.

What conception do mathematicians have of their science? What plan do they follow in their research? What principles direct their activity? What is the goal which they seek? These are the questions which the author of this book sets himself and to which he seeks an answer. Few probably will deny the value and importance of such questions nor the desirability that the devotees of mathematics should ask them of themselves and that they should have some clear notions as to the answers. In the vast complexity of modern mathematics, it is surely highly desirable that the man who is working in some particular corner and possibly on a very restricted problem should occasionally pause in his work in order to gain in proper perspective a view of the whole. Whither is he tending? What is his object? What constitutes real progress in mathematical investigations? What is important and why?

Although everyone will probably admit the importance of such considerations, it is to be feared that relatively few have seriously set themselves these questions. Still fewer perhaps have been able to formulate satisfactory replies. Indeed the present work shows how very difficult it is to answer the questions proposed.

The author sets himself a very definite problem. While his inquiry is in the nature of the case philosophical in the broad sense of the term, he is not at all concerned with the question of what place mathematics has in a general system of philosophy. He is concerned merely with the philosophy of mathematics as such. Nor is he concerned at all with the metaphysical aspects of the problem, but purely and simply with answers to the questions proposed. Furthermore he seeks an objective answer. He would eliminate as far as possible all personal bias and seek to discover answers to his questions in the actual work and progress of mathematics itself. His method therefore is, as he says, historical and critical. By a careful examination of the development of mathematics through the centuries he seeks to discover what the leading tendencies were and are. In a word, as indicated in the title of the work, what scientific ideals mathematicians of the past and present have set themselves. He is well aware of the difficulties of his problem and indeed there are many places in the book where the reader will be inclined to differ from the author in his placing of emphasis on such matters. There can be no doubt, however, that the author has written a very stimulating book which may be highly recommended to everyone interested in the questions discussed and should prove of special value to the young investigator starting upon his career and seeking orientation in his chosen field.

The author distinguishes three great epochs in the development of mathematics as bearing upon his problem: the epoch of Greek mathematics, the epoch beginning near the end of the 17th century and continuing for 150 years thereafter, and the present epoch beginning about the middle of the last century. The ideals governing the ancient Greeks were very largely æsthetic in character. They dealt with ideal concepts which had no concrete reality. The beauties they sought they considered to be inherent in the objects of study and not to be added to or subtracted from by the human intellect. Moreover, to satisfy their sense of the beautiful, their results had to be simple, harmonious. Another great ideal with which they furnished posterity was that of the geometric demonstration and of the deductive logical system. These ideals governed the development of mathematics through several centuries. They had inherent limitations, however, which the author analyzes with care. A geometric entity did not "exist" for them unless it could be "constructed," and quite obviously their ideals of simplicity and of the rôle that intuition should play in their discoveries involved serious limitations. Herein as well as in their contempt for practical applications may be sought the reasons why the Greeks never developed an algebra.

Preparation for the second epoch came through the introduction of algebra in the middle ages. In direct contrast to Greek ideals, its origin is to be found in practical applications. Algebra came to Western Europe in the form of practical rules for computation with practically no scientific foundation. Moreover progress depended largely on the absence of scientific scruples. The faith of the early investigators was superb. All through the 17th century and beyond, they were governed by the hope that they held in algebraic methods the mechanical key to all science. The new epoch as such may be said to begin with the publication of Descartes' *Geometry*, and with the invention of the calculus by Newton and Leibnitz. Descartes introduced a new conception in place of the euclidean demonstration, a new method which involved the discovery of geometric properties by indirection; that is, by the application of algebraic methods. The new conception of the character of mathematics introduced by the work of Descartes, Newton, and Leibnitz is the idea of synthesis, the idea of putting together simple elements in such a manner as to form progressively compounds of a more and more complicated character. Algebra is considered not as a collection of results but as a method of combination and discussion. In glancing back over the activities of the 150 years following the invention of the calculus, we can readily grasp the enthusiasm of the man working with his new tools. It is hardly an exaggeration to say that mathematical investigation at that time ceased to be a profession and became an industry. No limits were seen to the power of the new methods and all that seemed to be necessary was to proceed systematically in the building up of the edifice from its simple elements to more and more complicated and extensive structures. Leibnitz's dream of a general combinatory calculus whereby all problems of human thought should be capable of solution by an appropriate operational symbolism was a not unnatural consequence of the situation in which he found himself. This era of synthesis is then governed by the ideal that the perfect mathematical science is constructive and mechanical, the calculations of which are performed, so to speak, automatically.

The course of events, however, proved the limitations of the new methods. Blocked in its triumphant progress, it is only natural that a critical spirit should develop. The study of the logical foundations of the science became prominent. The author interprets even this phase of activity as belonging in part to the era of synthesis, the idea being that the axioms and postulates sought for were merely in order to provide a secure foundation for the edifice which had been and was being built. It would seem, however, to the reviewer, at least, that the development of postulational methods, especially in their latter course, belongs to the third epoch rather than to the second not merely chronologically, but also in spirit.

This third epoch is again sharply contrasted with the preceding in that it is characterized by analysis rather than by synthesis. The modern mathematician is like a chemist who analyzes an extremely complicated situation and seeks the elements of which it is compounded.* Our present epoch then is characterized by a frank recognition of the limitations of logic alone. Other intellectual activity than that of mere logic is necessary for further progress. Such activities are especially experimentation, the careful analysis of special cases, and above all the recognition of the power of intuition or insight. The modern mathematician must be constructive in the domain of ideas, not merely in the mechanical putting together of simple elements already existing. Progress at present demands the development of new points of view for classifying and interpreting the baffling new problems which present themselves.

In his final chapter on the present mission of mathematics, the author attempts to appraise the manifold and apparently conflicting tendencies that are at present in existence. An extended and interesting discussion of the relation of mathematics to theoretical physics leads to the rather obvious conclusion that the demands of the applications of mathematics cannot furnish the sole or even the principal guide to further progress. The author then takes up the claims of those who would find the desired guide post in the æsthetic or artistic element in mathematics, only to reject this also. He admits that this orientation of our science has been fruitful in that it has served to introduce a large number of new ideas. But, he says, it merely raises the fundamental question in another form: "What precisely does the mathematician mean by 'beautiful,' 'elegant,' 'remarkable?'"

The author thus admits himself unable to give any satisfactory answer to the questions he has set himself. He has recourse finally to the advice often given to the young aspirant toward mathematical research: Study the great masters! They had a certain flair for recognizing the valuable and important directions of advance. And the fact remains that, even though we cannot find any simple rules which govern the directions of progress, progress does exist. Our science has advanced and is continually advancing in spite of the lack of any conscious direction.

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* It may be desirable at this point to caution the reader against a possible confusion of terms. The text of our review would seem to make sufficiently clear the sense in which we (and the author) are using the words "synthesis" and "analysis." The possible confusion arises from the fact that the method used in the second epoch and which would seem to be essentially synthetic in character has received the name of "mathematical analysis."

The author obviously laid down his pen after writing the last word of his interesting little book with a feeling of discouragement and dissatisfaction. The reader shares this feeling;—but, in spite of it, he feels that the writing and the reading has been worth while. The questions raised are of fundamental importance and of the greatest interest. The fact that they remain to a large extent unanswered is merely a challenge to the future. The reviewer has a feeling that the answer may possibly be found in a more vigorous attack on the question which the author himself raises but which he dismisses with a few words. Just what is implied by the words "beautiful," "elegant," "remarkable" as used by the mathematician? Just what is the "flair" which the great masters possess? Is it not possible that this flair is essentially artistic in its nature and that the development of mathematical science is governed largely by laws analogous to those that govern the development of the fine arts?

J. W. YOUNG

SHORTER NOTICES

Ueber Spiralen. By Archimedes. Translated and annotated by Arthur Czwalina-Allenstein. Leipzig, Akademische Verlagsgesellschaft, 1922. 71 pp.

This German translation of Archimedes's classic work on spirals (Ostwald's Klassiker, No. 201), which is now published on account of the fact that Nizze's German translation of 1824 has long been out of print, is of no significance for the American student, as we have Heath's admirable translation.* The supplement (pp. 61–71) gives a reconstruction of a possible method by which Archimedes may have been led to his results; the method is ingenious and plausible, but it has the serious defect that it considers the ratio of an area to a volume, which would have been anathema to a Greek of the classical period; so that we can hardly be convinced, in the absence of evidence, that even so original a genius as Archimedes would have hit upon this particular method. It is probably as well to confess that we are entirely ignorant of the way in which Archimedes did arrive at his admirable results.

R. B. MCCLENON

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* *The Works of Archimedes,* edited in modern notation with introductory chapters by T. L. Heath. Cambridge, 1897.

Einführung in die Ebenen und Körperlichen Oerter. By Pierre de Fermat. Translated and edited by H. Wieleitner. Leipzig, Akademische Verlagsgesellschaft, 1923. 22 pp.

This little volume of the well known Ostwald's Klassiker series is even smaller than the pagination would indicate, as the preface occupies pages 5–6, the translation of Fermat's memoir pages 7–17, and the notes pages 18–22. Fermat, like Descartes and his other contemporaries, follows the Greek nomenclature, "plane" loci meaning straight lines and circles, and "solid" loci meaning conics. Curves of higher degree were called "linear." In this essay the various forms which equations of the first or second degree in two variables can take are briefly discussed, and it is shown geometrically that such equations lead to either "plane" or "solid" loci. Besides its interest to the student of the history of mathematics, the volume may be recommended to undergraduates who are working with the general equation of the second degree, as a striking illustration of the gain in power which they obtain through the use of analytic methods. It seems to the reviewer that some of Fermat's tangent constructions and problems in maxima and minima should have been included, thus rounding out one important section of Fermat's mathematical work, and incidentally bringing the book up to a size comparable with the others of the series.

R. B. McClenon

Theorie und Anwendung der Unendlichen Reihen. By K. Knopp. Berlin, J. Springer, 1922. 10 + 474 pp.

Occasionally one finds an individual who has the ability to present mathematical ideas in lectures in a perfectly translucent elegant manner. It is a real pleasure to listen to such a man, and to watch him develop the subject in a skilful way. It is an ideal towards which every one who lectures on mathematics aspires, but which seems so difficult of attainment. Of the same degree of rarity is the man who has the ability to write as though he were presenting the subject matter to a group of individuals, and who does it in such a delightfully clear way that it is a real joy to read and to absorb. Such seems to be the gift of the author of this volume on infinite series.

The book is not intended to give a profound discussion of the subject. That has recently been done by Pringsheim. Its purpose seems rather to be to provide an introduction to some of the fundamental ideas of the theory of functions of a real variable by showing their application in the theory of limits and particularly infinite series and products. The gap between a course in the calculus, and a profound discussion of the theory of functions of a real variable is often very hard to bridge. The study of infinite series in the manner indicated in this book would be an admirable way of doing this very thing.

A brief survey of the contents will indicate the scope of the book. The first part is devoted to a discussion of the irrational number, leading to a definition of limit of a sequence, the fundamental operations and properties of rational number being assumed. Then follows an elementary discussion of convergence of infinite series, leading to a treatment of power series, and the expansions of the elementary functions in power series. Infinite products are treated in the same elementary way. Next comes a more extensive study of infinite series, containing a brief survey of the present situation as far as the convergence and divergence of series is concerned. A chapter on series of functions leads naturally to a brief discussion of Fourier series. Similarly, the treatment of series of complex numbers leads to the mention of Dirichlet and factorial series. The book closes with a chapter on divergent series. Problems are provided at the close of each chapter, to lead the reader into a deeper consideration of the subject.

There is much of value in this book for the person who wants an introduction to the study of infinite series, and the person who wants a survey and derivation of much scattered material of interest in this field. Altogether it is a book very much to be recommended to the person who is making his acquaintance with higher mathematics along analysis lines.

T. H. HILDEBRANDT

Early Science in Oxford. Part II. *Mathematics.* By R. T. Gunther. London, Oxford University Press, 1922. 101 pp., 10 plates, colored frontispiece.

The early history of American mathematics is most intimately linked with the development of mathematical science in England. Not only the terminology, inevitably, but equally the content of our early text-books was based directly upon the English models which themselves served as texts in the period from 1650 to 1750 or even later. For this reason details concerning early Oxford, so long the center of mathematics in England, are of particular interest to us.

The illustrations of this work are fine and well chosen; the lists of the early mathematicians and astronomers of Oxford are a welcome addition to the literature of the subject. Unfortunately the text (pp. 1–33) reveals an amazing lack of familiarity with recent publications on the history of mathematics, abounding in errors which could have been avoided by reference to books on the shelves of the Bodleian. However, the descriptive catalogue of early mathematical instruments belonging to the University and Colleges of Oxford, the real *raison d'être* of the work in question, is made with great ability and care. This material constitutes a most worthy addition to our knowledge of mathematical instruments which played so large a part in the discovery and exploration period of American history.

It is not worth while to enumerate the errors on pages 1–33. The statements concerning the sources of Recorde's text-books illustrate the careless editing of the work. Recorde's *Ground of Arts* was not "based on manuscripts entitled *De origine artium* and *Arithmeticae principia*" nor was his *Pathway* (sic) *to Knowledge* "printed from manuscripts entitled *Geometriae semita* and *Theoremata Geometriae*." Neither was the "Whetstone of Witte" "printed . . . from his Secunda pars Arithmeticae." These works were all in English and the material is that current in continental texts of the same period. In passing, it must be said that Recorde was neither one of the "earliest" nor yet one of the "greatest mathematicians" of England.

In spite of the introduction which should have been prepared by one more familiar with the history of mathematics the work is highly to be commended, as eminently worthy a place in mathematical libraries.

Invitation is extended to subscribe to a guarantee fund to cover the cost of plates for Part III, Astronomy. American libraries are urged to send subscriptions to Part III to R. T. Gunther at Magdalen College, Oxford.

L. C. Karpinski

Geschichte der Elementar-Mathematik in systematischer Darstellung. By Johannes Tropfke. Berlin and Leipzig, Vereinigung Wissenschaftlicher Verleger. Bd. I: *Rechnen.* 1921. vi + 177 pp. Bd. II: *Allgemeine Arithmetik.* 1921. 221 pp. Bd. III: *Proportionen. Gleichungen.* 1922. 151 pp. Bd. IV: *Ebene Geometrie.* 1923. 238 pp.

The first edition of Tropfke's *Geschichte der Elementar-Mathematik* was published in 1902–1903, and was reviewed by J. W. A. Young in this Bulletin.* The four volumes under review form part of a second edition, which is to include seven volumes, the last three being yet in press, or at least not yet available in this country. That the revision has been thoroughgoing is evidenced by the fact that the material corresponding to these four volumes occupied approximately 510 pages in the first edition, and has thus been expanded above 50 per cent; while the references to the literature in these four volumes number 4348, as compared with 1951 in the corresponding parts of the first edition.

And the advance made beyond the first edition is by no means merely quantitative. The author was fortunate in having the active assistance of G. Eneström and H. Wieleitner in the preparation of the new edition, and their names are a sufficient guarantee that no pains have been spared to make the work as complete and authoritative as possible. On nearly every page we find valuable additions to the information given in the first edition; while in several cases the point of view then adopted has been radically changed or even reversed.

To mention but a few of the changes: (1) In discussing the origins of the number system, use has been made of recent researches as to the knowledge of the Babylonians. An interesting detail is the fact (I, p. 15) that as early as 2500 B. C. the tables of Senkereh contain representations of very large numbers according to the sexagesimal system, the largest to date deciphered being $60^8 + 10 \cdot 60^7$ ($= 195\,955\,200\,000\,000$). (2) The history of the development of technical terms is very largely expanded. (3) The independence of the work of the Hindus is in many cases questioned or denied in the second edition where it was accepted in the first; the researches of G. R. Kaye are largely responsible for the author's change of view here. (4) The account of complex numbers (II, 79–90) is entirely rewritten and considerably enlarged. (5) The discussion of the development of the theory of parallels (IV, 53–60) is a considerable improvement over that in the first edition, while it must be confessed that it still leaves something to be desired; thus Saccheri is only referred to in a footnote, and Lambert is not mentioned at all. (6) The work of several individuals has evidently been more thoroughly investigated; thus Leonardo of Pisa's monumental work is more adequately described than in the first edition, while at the same time further information is given as to his Arabian sources. Apparently, however, Abu Kamil has been overlooked among these.* (7) The discussion of indeterminate equations is expanded from 10 to 15 pages (III, 99–114). Here, as well as under the topic "properties of integers" (I, 93–110), we miss any reference to Dickson's *History of the Theory of Numbers*, which would have been of great value to the author, but which presumably was not available during the work of revision.

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* Vol. 12 (1905), pp. 138–140.

Two topics which the reviewer would have been glad to see included but which do not seem to be mentioned are the "regula falsi," and the so-called "Russian peasant method" of multiplication. A temporary drawback is the lack of an index, as that is to appear in the last volume. This is of itself a sufficient reason why the publication of the remainder of the work will be awaited with impatience.

A few minor slips or misprints were noted: In referring to the use of the word "cipher" for "zero" (I, 9–10) the author implies that this use is obsolete in English. The date of Leonardo of Pisa's birth is not known to be 1180 (III, 121). The credit for the proof of Steiner's Malfatti problem construction is given to Schroeter, 1874 (IV, 126), whereas it should be given to A. S. Hart, who published a proof in the QUARTERLY JOURNAL OF MATHEMATICS in 1857 (vol. I, pp. 219–221). Misprints occur several times as to dates; thus Albert Girard died in 1632, and the date is so printed in volume I, page 7, but thereafter 1633 is several times found. (II, 76, III, 95, and IV, 123, for example.) Roger Cotes was born in 1682, instead of 1652, as stated on page 159 of volume II. The correct date is given at page 213 of the same volume. There is a misprint on page 134 of volume II, referring to the "fragments of Kahun," which owing to the lack of an index is not easy to correct. The first reference in the "Zeittafel," volume III, page 115, should be I, 128, instead of I, 24. (I, 24 is correct if referring to Leonardo of Pisa's *Scritti*.)

It would be desirable that a reference work of this sort, so near to complete accuracy, should be made even more valuable and kept up to date by the publication at intervals of additions and corrections which might be contributed by anyone interested; possibly the AMERICAN MATHEMATICAL MONTHLY might serve as the medium for such publication.

Tropfke's style is clear and lucid, enlivened by imagery where this can be used without detracting from accuracy. Biographical details are entirely omitted; had they not been, the size of the work must have been expanded to impossible proportions. The topical arrangement brings advantages which fully compensate for the lack of unity which it necessitates. From every point of view the revised edition must be pronounced a decided success, so far as the four volumes permit a judgment of the whole. It can be recommended most warmly to everyone who is interested in the history of elementary mathematics.

R. B. MCCLENON

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* Cf. L. C. Karpinski, *The Algebra of Abu Kamil*, AMERICAN MATHEMATICAL MONTHLY, vol. 21 (1914), pp. 37–48.

Einführung in die Mechanik Deformierbarer Körper. By Max Planck. Zweite Auflage. Leipzig, Hirzel, 1922. 191 pp.

The first edition of this little book appeared in 1919. This second edition is little more than a corrected first edition. The contents are divided into three parts. Part one (two chapters) deals with strains and stresses in an elastic medium. Part two (four chapters) deals with infinitesimal deformations and in particular with the relations connecting the coefficients of strain and stress. In the last two chapters the preceding theory is applied to the problem of the vibration of solids and fluids. Part three (four chapters) is devoted to rotational and irrotational fluid motion. The last chapter is on viscosity. The equations of hydrodynamics are derived as special cases of the general equations of elasticity. The equation of continuity is given in two forms, one stating that a given small mass of fluid remains invariant, and the other that the difference between the inflow and outflow from a small volume is equal to the increase in the mass contained in the small volume.

The author being a physicist, the treatment is more physical than the usual book on elasticity and hydrodynamics, with the exception of that by Thompson and Tait The style is clear, concise, and appealing. Even one familiar with the subject will find Planck's treatment refreshing and fascinating.

C. L. E. Moore

Lehrbuch der darstellenden Geometrie für technische Hochschulen. By Dr. Emil Müller. Zweiter Band, dritte Auflage. Leipzig and Berlin, B. G. Teubner, 1923. x + 362 pp.

Vorlesungen über darstellende Geometrie. By Dr. Emil Müller. Band I. *Die linearen Abbildungen.* Revised by Dr. Erwin Kruppa. Leipzig and Vienna, Franz Deuticke, 1923. xi + 292 pp.

The first edition of Professor Müller's text on descriptive geometry was reviewed by Professor Snyder [vol. I (1908), 1st part of vol. II (1912), and 2d part of vol. II (1916) in this Bulletin, vol. 16, p. 136; vol. 20, p. 253; and vol. 24, p. 257]. The second editions of the two parts of the second volume, which appeared in 1919, were practically the same as the first edition. In the preface to the third edition of volume II (where the two parts are bound together), the author says that he has made only slight changes, although he has much new material which he might have inserted.

In the preface to the Müller-Kruppa book, Professor Müller says that soon after he took the chair of descriptive geometry (in 1902) at the Technical High School of Vienna, he began to supplement his regular teaching by advanced lectures containing some results of his own researches. These gradually settled down to a four-year cycle, although the content has naturally undergone changes. As he did not have the time to get any of this material into shape for publication, he was happy when his former pupil and present colleague was willing to undertake the task for some of the lectures. Although Professor Kruppa had the original manuscripts, Professor Müller gives him the credit for the final wording, for the drawings, for the revision of some parts of the manuscript, and for certain additions.

This volume consists of three parts: I, Central projection; II, The principles of linear representation; III, Special representations. It also contains a subject index and a name index. In the second part the three principles are called "Zweispuren," "Zweibilder," and "achsonometrische." Ten pages are devoted to Pohlke's theorem. In the third part the four chapters deal with the ordinary linear representations, relief perspective, a representation of plane motion in point space (kinematic representation of Blaschke and Grünwald), and a projective generalization of Lie's line-sphere transformation. Four of the ten chapters of the book are concluded by lists of exercises.

This volume is not intended for beginners: it presupposes a knowledge of both descriptive and projective geometry. But for one who is prepared to read it, it offers an excellent treatment of interesting and valuable material.

E. B. Cowley

Methodik des mathematischen Unterrichts. 2te durchgesehene und vermehrte Auflage. Zweiter Teil: Didaktik der einzelnen Gebiete. By W. Lietzmann. Leipzig, Quelle und Meyer, 1923. xii + 367 pp.

This work covers the customary mathematical subjects from arithmetic to analytic geometry, the elements of modern geometry, and of calculus, inclusive. In an easy, almost chatty style, the author discusses, at will, points of subject matter, of method and of twentieth century developments in the teaching of the topic in hand. References to twentieth century publications are liberally furnished, to German works as a rule, but also to some in French and a few in English. From arithmetic to analytic geometry, there are given numerous pictures, often from photographs, illustrating models and apparatus. Mathematical moving pictures are discussed (pp. 126–8), and the upshot of the matter is that films have not yet established themselves as valuable adjuncts to mathematical instruction. The illustrations must speak for themselves; no description of the manner of construction or *modus operandi* is given. Nevertheless, for the average American reader these illustrations will perhaps be the most interesting and suggestive portion of the book. They certainly are the most easily accessible.

J. W. A. Young

La Relativité Vraie et la Gravitation Universelle. By Georges Fournier. Paris, Gauthier-Villars, 1923. viii + 130 pp.

This alleged refutation of the work of Einstein must be classed with the work of the circle-squarers. Since those teaching the theory of relativity might use this pamphlet as material for an exercise in locating fallacies, I will not list them here.

It is disappointing to find the imprint of Gauthier-Villars on such a book. This is not the only recently published French work which has suggested the intrusion of a spirit of nationalism into the scientific world.

C. N. Reynolds, Jr.

Weber-Wellstein, Encyklopädie der Elementarmathematik. Erster Band: *Arithmetik, Algebra und Analysis.* By Heinrich Weber. Vierte Auflage neubearbeitet von Paul Epstein. Leipzig und Berlin, B. G. Teubner, 1922. xvi + 568 pp.

The first edition of this standard and valuable work was reviewed in this BULLETIN (vol. 10, p. 200) by David Eugene Smith. A brief notice of the third edition was published by F. W. Owens (BULLETIN, vol. 17, p. 546). Since the publication of the latter, both the original authors have died. The new (fourth) edition of the first volume appears under the editorship of Paul Epstein. A critical estimate of such a standard work is unnecessary, especially so since no radical changes appear to have taken place. The most noteworthy is the omission of the short chapter on differentiation, which was introduced first in the second edition, and which is now again abandoned partly to save space for material that appears to the present editor more important and partly because a mere brief introduction is not in keeping with plan of the work. An extended treatment was in any case out of the question. Opinions will probably differ as to the validity of the reasons assigned for this omission; but to the present reviewer they appear altogether reasonable. The question as to the meaning to be attached to the term "elementary mathematics" is in any case a difficult one. The meaning accepted by the present author seems to be at least as reasonable as any other. The present edition is in spite of the omission noted somewhat larger than the previous one, owing to the inclusion of some new material and the elaboration of some of the older material. The preface to the present edition indicates these additions, so that they need not be listed here. The historical and bibliographical notes seem to have been distinctly improved. There can be no doubt that the new edition will adequately fill the place in our standard reference literature which the earlier editions filled with such marked success.

J. W. YOUNG

Über die Hypothesen welche der Geometrie zu Grunde liegen. By B. Riemann. Neu herausgegeben und erläutert von H. Weyl. Second edition. Berlin, Julius Springer, 1921. vi + 47 pp.

This reprint of Riemann's famous *Habilitationsschrift* will be welcomed by many. It is one of those few papers which are of permanent and fundamental interest to all who are concerned with the consideration of the general foundation and formulation of fundamental concepts. It is of special value at the present time when the theory of relativity is demanding so much attention. In the last section of his paper Riemann exhibits an almost uncanny insight which Einstein's results finally corroborate. Weyl's notes are very helpful, not only in carrying out in some detail a number of the analytic computations which in Riemann's text are only indicated, but also in giving significant references to later related literature. These notes occupy twenty-four of the forty-seven pages of the pamphlet.

J. W. YOUNG

NOTES

The second number of vol. 24 of the TRANSACTIONS OF THIS SOCIETY (September, 1922) contains: *Generalized limits in general analysis, first paper*, by C. N. Moore; *Anharmonic polynomial generalizations of the numbers of Bernoulli and Euler*, by E. T. Bell; *New properties of all real functions*, by Henry Blumberg; *A fundamental system of invariants of a modular group of transformations*, by J. S. Turner; *The Gaussian law of error for any number of variables*, by J. L. Coolidge; *Certain theorems relating to plane connected point sets*, by Anna M. Mullikin.

Professor Edward Kasner has been elected a member of the Editorial Committee of the TRANSACTIONS OF THIS SOCIETY, as successor to Professor L. P. Eisenhart. Papers in geometry offered to the TRANSACTIONS should be sent to Professor Kasner. Professor Olive C. Hazlett and Dr. Einar Hille have been appointed associate editors.

During the suspension of publication of the BIBLIOTHECA MATHEMATICA, corrections to Cantor's *Vorlesungen über Geschichte der Mathematik* will appear in the JAHRESBERICHT DER DEUTSCHEN MATHEMATIKER-VEREINIGUNG.

The Vienna Academy of Sciences offers a prize for a memoir on *Fictions in mathematics*, at the suggestion of the Society of Friends of the Philosophy of "Als Ob," which has placed the sum of one million marks at the disposal of the Academy for this purpose. Competing memoirs should be written in German, and should be sent to the Academy before December 31, 1925.

Dr. Mallison, Master of Clare College, Cambridge, has offered £500 for a prize, to be called the Mayhew Prize, to be awarded by examination in Part II of the (Cambridge) Mathematical Tripos, to the candidate who in the opinion of the examiners is of greatest merit, preference being given, other things being equal, to candidates whose main subjects are in branches of applied mathematics.

Professors A. Kneser, of the University of Breslau, and E. Study, of the University of Bonn, have been elected corresponding members of the Prussian Academy of Sciences.

Professor E. Strömgren, of the University of Copenhagen, has been called to a professorship of theoretical astronomy at the University of Berlin.

Professor G. Hergoltz, of the University of Leipzig, has been called to a professorship at the University of Munich.

Professor E. Hilb has been promoted to a full professorship at the University of Würzburg.

Dr. F. Böhm has been appointed to an associate professorship at the University of Munich.

Dr. K. Löwner has been admitted as privat docent in mathematics at the University of Berlin.

Professor Niels Bohr, of the University of Copenhagen, will deliver the Silliman lectures at Yale University, and also the Simpson lectures at Amherst College during the present academic year.

Professor J. M. Willard, head of the department of mathematics at Pennsylvania State College, has retired from active service.

Professor F. N. Cole, of Columbia University, ex-secretary of the American Mathematical Society, has been granted leave of absence for the second half of the present academic year.

Dr. Irwin Roman, of Northwestern University, has been appointed associate professor of mathematics at Vanderbilt University.

Mr. A. D. Campbell, of Cornell University, has been appointed assistant professor of mathematics at the University of Arkansas.

Professor O. W. Albert, of Grinnell College, has been appointed head of the department of mathematics at the University of Redlands.

Mr. D. L. Holl, of the University of Chicago, has been appointed assistant professor of mathematics at Ohio Wesleyan University.

Dr. G. M. Robison, of Cornell University, has been appointed assistant professor of mathematics at Trinity College, Durham.

Dr. G. E. Raynor, of Princeton University, has been appointed assistant professor of mathematics at Wesleyan University.

At Princeton University, Dr. C. E. Hille and Dr. C. C. MacDuffee have been promoted to assistant professorships of mathematics.

The two Benjamin Peirce Instructorships in Mathematics at Harvard University (see this Bulletin, vol. 21, page 315) are again open to general competition. Applications for the year 1924–25, accompanied by the necessary papers, should reach Professor Birkhoff not later than February 15, 1924, and all inquiries relating to these appointments should be addressed to him.

Professor J. A. Cragwall, head of the department of mathematics at Wabash College, has been granted a year's leave of absence because of poor health. His duties are being temporarily taken over by Professor G. E. Carscallen.

Professor L. Krüger, formerly director of the Geodetic Institute at Potsdam, died June 1, 1923, at the age of sixty-five years.

Professor Oskar von Peinthner-Lichtenfels, of the Technical School of Graz, died June 9, 1923.

Professor C. N. Little, dean of the college of engineering of the University of Idaho, died September 7, 1923. Professor Little was known for his contributions to the theory of knots.

Professor D. T. Wilson, of the department of astronomy of Case School of Applied Science, died October 12, 1923, in Washington, D. C.

Charles Proteus Steinmetz, professor of electrophysics in Union College and chief consulting engineer for the General Electric Company, died in Schenectady on October 26, 1923, from heart failure, at the age of fifty-eight years. He had been a member of the American Mathematical Society since 1891.

NEW PUBLICATIONS

PART I. PURE MATHEMATICS

ARCHIMEDES. Die Quadratur der Parabel. Ueber das Gleichgewicht ebener Flächen oder über den Schwerpunkt ebener Flächen. (Ostwald's Klassiker der Exakten Wissenschaften.) Leipzig, Akademische Verlagsgesellschaft, 1923. 64 pp.

BESSEL-HAGEN (E.). See KLEIN (F.).

BIANCHI (L.). Lezioni sulla teoria dei numeri algebrici. Pisa, Spoerri, 1923. 6 + 641 pp.

DESCARTES (R.). Geometrie. Deutsch herausgegeben von L. Schlesinger. 2te, durchgesehene Auflage. Leipzig, Mayer und Müller, 1923. 11 + 121 pp.

FRANK (E.). Plato und die sogenannten Pythogoreer. Ein Kapitel aus der Geschichte des griechischen Geistes. Halle, Verlag von Max Niemeyer, 1923. 10+ 400 pp.

FRICKE (R.). See KLEIN (F.).

GILLMER (M.). Trigonometrie und Stereometrie. Leipzig, Hirzel, 1922. 6 + 226 pp.

IMBODEN (A.). Resumen de geometria proyectiva elemental. Paraná, Prometeo, 1923. 22 pp.

KLEIN (F.). Gesammelte mathematische Abhandlungen. Band 3: Elliptische Funktionen, insbesondere Modulfunktionen, Hyperelliptische und abelsche Funktionen, Riemannsche Funktionentheorie und automorphe Funktionen. Anhang: Verschiedene Verzeichnisse. Herausgegeben von R. Fricke, H. Vermeil, und E. Bessel-Hagen. Berlin, Springer, 1923. 10 + 774 + 36 pp.

LEVY (H.). See RUNGE (C.).

✓LOVE (C. E.). Analytic geometry. New York, Macmillan, 1923. 14 + 306 pp.

VON MANGOLDT (H.). Einführung in die höhere Mathematik. 4te Auflage. Band I. Leipzig, Hirzel, 1923.

MAYER (O.). Contributions à la théorie des quartiques bicirculaires. (Thèse, Jassy.) Jassy, Imprimerie H. Goldner, 1923. 99 pp.

NEYMEYER (L.). Zur Theorie der Krümmungsmasse mehrdimensionaler Mannigfaltigkeiten. (Diss.) Freiburg i. Br., 1922.

ONNEN (H.). Kreisevolventen und algebraische Funktionen. Leipzig, Teubner, 1923. 43 pp.

PASCALI (J.). Calcolo de diferencias finitas. Buenos Aires, Revista del Centro de Estudiantes de Ingenieria, 1922. 16 pp.

PICARD (E.). La vie et l'œuvre de Pierre Duhem. Paris, Gauthier-Villars, 1922. 4to. 58 pp.

RUNGE (C.). Vector analysis. Translated by H. Levy. London, Methuen, 1923. 8 + 226 pp.

SCHLESINGER (L.). See DESCARTES (R.).

SCHRUTKA (L.). Zahlenrechnung. Leipzig, Teubner, 1923. 10 + 146 pp.

VERMEIL (H.). See KLEIN (F.).

PART II. APPLIED MATHEMATICS

ANDRADE (E.). The structure of the atom. London, Bell, 1923. 15 + 314 pp.

BIDWELL (C. C.). Outlines of physics. Ithaca, Comstock Publishing Company, 1922. 104 pp.

BOND (P. S.). Map reading and military sketching. New York, Army and Navy Journal, 1922. 104 pp.

BOUASSE (H.). Dynamique générale. Paris, Delagrave, 1923. 324 pp.

BRILLOUIN (H.). Théorie électrique moderne de l'état solide. Paris, Société française des Electriciens, 1923. 8vo. 24 pp.

CODD (L. W.). See NERNST (W.).

DINGLER (H.). Das Problem des absoluten Raumes. Leipzig, Hirzel, 1923. 50 pp.

ENCYCLOPÄDIE der mathematischen Wissenschaften. Band V 2, Heft 5: R. Seeliger, Elektronentheorie der Metalle. Leipzig, Teubner, 1922.

HOPKINS (M.). Chance and error. The theory of evolution. New York, Dutton, and London, Kegan Paul, Trench, Trübner, 1923. 8 + 223 pp.

KADERAVEK (F.). Perspektiva. Prague, J. Stene, 1922. 109 pp. + 31 tables.

KIRBY (R. S.). Exercises in the elements of descriptive geometry. New York, Wiley, 1922. 10 + 49 pp.

KOBOLD (H.). See MÖBIUS (A. F.).

LARNER (E. T.). Alternating currents, their theory and transmission. London, Lockwood, 1922. 8 + 198 pp.

VON LAUE (M.). Ueber die Auffindung der Röntgenstrahlinterferenzen. Nobelvortrag. Karlsruhe, C. F. Müller, 1920. 16 pp.

LEWIS (G. M.) and RANDALL (M.). Thermodynamics and free energy of chemical substances. London, McGraw-Hill, 1923. 23 + 653 pp.

MARAIS (H.). Introduction géométrique à l'étude de la relativité. Paris, Gauthier-Villars, 1923. 8vo. 192 pp.

MIE (G.). Die Einstein'sche Gravitationstheorie. 2te Auflage. Leipzig, Hirzel, 1923. 4 + 69 pp.

MÖBIUS (A. F.). Astronomie. Grösse, Bewegung und Entfernung der Himmels-Körper. 13te Auflage, bearbeitet von H. Kobold. Teil 2. Berlin, Vereinigung wissenschaftlicher Verleger, 1923. 128 pp.

MOREUX (T.). La science mystérieuse des Pharaons. Paris, Doin, 1923. 16mo. 250 pp.

NERNST (W.). Theoretical chemistry from the standpoint of Avogadro's rule and thermodynamics. 5th edition, revised by L. W. Codd. London, Macmillan, 1923. 20 + 922 pp.

OLLIVIER (H.). Cours de physique générale. Tome III. 2e édition, entièrement refondue. Paris, Hermann, 1923. 712 pp

RANDALL (M.). See LEWIS (G. M.).

SEELIGER (R.). See ENCYKLOPÄDIE.

THIRTY-SECOND ANNUAL LIST OF PAPERS

READ BEFORE THE AMERICAN MATHEMATICAL SOCIETY AND SUBSEQUENTLY PUBLISHED, INCLUDING REFERENCES TO THE PLACES OF PUBLICATION

Alexander, J. W. A proof and extension of the Jordan-Brouwer separation theorem. Read April 29, 1916. *Transactions of this Society*, vol. 23, No. 4, pp. 333–349; June, 1922.

—— Invariant points of a surface transformation of given class. Read Dec. 28, 1922. *Transactions of this Society*, vol. 25, No. 2, pp. 173–184; April, 1923.

Barnett, I. A. Differential equations with a continuous infinitude of variables. Read Dec. 28, 1918. *American Journal of Mathematics*, vol. 44, No. 3, pp. 172–190; July, 1922.

—— Linear partial differential equations with a continuous infinitude of variables. Read Dec. 28, 1918, and April 24, 1920. *American Journal of Mathematics*, vol. 45, No. 1, pp. 42–53; Jan., 1923.

Bell, E. T. On restricted systems of higher indeterminate equations. Read (San Francisco) June 18, 1920. *Transactions of this Society*, vol. 22, No. 4, pp. 483–488; Oct., 1921.

—— Anharmonic polynomial generalizations of the numbers of Bernoulli and Euler. Read (San Francisco) April 9, 1921. *Transactions of this Society*, vol. 24, No. 2, pp. 89–112; Sept., 1922.

—— Periodicities in the theory of partitions. Read (San Francisco) April 8, 1922. *Annals of Mathematics*, (2), vol. 24, No. 1, pp. 1–22; Sept., 1922.

—— Relations between the numbers of Bernoulli, Euler, Genocchi, and Lucas. Read (San Francisco) April 8, 1922. *Messenger of Mathematics*, vol. 52, No. 4, pp. 56–64, and No. 5, pp. 65–68; Aug. and Sept., 1922.

—— Euler algebra. Read (San Francisco) Oct. 22, 1921, and April 7, 1923. *Transactions of this Society*, vol. 25, No. 1, pp. 135–154; Jan., 1923.

—— Singly infinite class number relations. Read (San Francisco) Oct. 23, 1920. *Quarterly Journal of Mathematics*, vol. 49, No. 4, pp. 322–337; March, 1923.

—— Applications of analysis to the arithmetic of higher forms. Read (San Francisco) Oct. 21, 1922. *Transactions of this Society*, vol. 25, No. 2, pp. 185–189; April, 1923.

—— Square-partition congruences. Read (San Francisco) April 7, 1923. *This Bulletin*, vol. 29, No. 8, pp. 349–355; Oct., 1923.

—— Analogies between the u_n, v_n of Lucas and elliptic functions. Read (San Francisco) Sept. 18, 1923. *This Bulletin*, vol. 29, No. 9, pp. 401–406; Nov., 1923.

Bennett, A. A. Normalized geometric systems. Read Oct. 30, 1920. *Proceedings of the National Academy of Sciences*, vol. 7, No. 3, pp. 84–89; March, 1921.

—— Some algebraic analogies in matric theory. Read Feb. 28, 1920. *Annals of Mathematics*, (2), vol. 23, No. 1, pp. 91–96; Sept., 1921.

—— Tables for interior ballistics. Read Feb. 26, 1921. *Ordnance Department Document*, No. 2039. Washington, Government Printing Office, 1923. 80 pp.

Bernstein, B. A. On the complete independence of Hurwitz's postulates for abelian groups and fields. Read (San Francisco) Oct. 22, 1921, *Annals of Mathematics*, (2), vol. 23, No. 4, pp. 313–316; June, 1922.

Birkhoff, G. D. Circular plates of variable thickness. Read Sept. 8, 1921. *Philosophical Magazine*, ser. 6, vol. 43, No. 5, pp. 953–962; May, 1922.

Birkhoff, G. D., and Langer, R. E. The boundary problems and developments associated with a system of ordinary linear differential equations of the first order. Read Feb. 25, 1922. *Proceedings of the American Academy of Arts and Sciences*, vol. 58, No. 2, pp. 51–128; April, 1923.

Bliss, G. A. The reduction of singularities of plane curves by birational transformation. Read Dec. 28, 1922. *This Bulletin*, vol. 29, No. 4, pp. 161–183; April, 1923.

Blumberg, H. New properties of all real functions. Read March 29, 1919, and Dec. 30, 1920. *Transactions of this Society*, vol. 24, No. 2, pp. 113–128; Sept., 1922.

Bray, H. E. Proof of a formula for an area. Read Dec. 27, 1922. *This Bulletin*, vol. 29, No. 6, pp. 264–270; June, 1923.

Brinkmann (H. W.). On Riemann spaces conformal to euclidean space. Read Dec. 27, 1922. *Proceedings of the National Academy of Sciences*, vol. 9, No. 1, pp. 1–3; Jan., 1923.

—— On Riemann spaces conformal to Einstein spaces. Read April 28, 1923. *Proceedings of the National Academy of Sciences*, vol. 9, No. 5, pp. 172–174; May, 1923.

Buchanan, D. Asymptotic planetoids. Read Dec. 31, 1919. *Transactions of this Society*, vol. 23, No. 4, pp. 409–431; June, 1922.

Bussey, W. H. A note on the problem of eight queens. Read March 28, 1919. *American Mathematical Monthly*, vol. 29, No. 7, pp. 252–253; Aug., 1922.

Cajori, F. Origin of the names arithmetical and geometrical progression and proportion. Read (San Francisco) Oct. 21, 1922. *School Science and Mathematics*, vol. 22, No. 8, pp. 734–737; Nov., 1922.

—— Mathematical signs of equality. Read Dec. 27, 1922. *Isis*, vol. 5, No. 1, pp. 116–125; 1923.

—— The origin of the symbols for "degrees, minutes, and seconds." Read (San Francisco) Oct. 21, 1922. *American Mathematical Monthly*, vol. 30, No. 2, pp. 65–66; Feb., 1923.

—— Recent symbolisms for decimal fractions. Read (San Francisco) April 8, 1922. *Mathematics Teacher*, vol. 16, No. 3, pp. 183–187; March, 1923.

—— Varieties of minus signs. Read (San Francisco) April 7, 1923. *Mathematics Teacher*, vol. 16, No. 5, pp. 295–301; May, 1923.

—— The evolution of our exponential notation. Read (San Francisco) April 7, 1923. *School Science and Mathematics*, vol. 23, No. 6, pp. 573–581; June, 1923.

—— Grafting of the theory of limits on the calculus of Leibniz. Read Dec. 28, 1922. *American Mathematical Monthly*, vol. 30, No. 5, pp. 223–234; July–Aug., 1923.

Camp, C. C. Expansions in terms of solutions of partial differential equations. First paper: Multiple Fourier series expansions. Read Sept. 7, 1922. *Transactions of this Society*, vol. 25, No. 1, pp. 123–134; Jan., 1923.

Candy, A. L. Cyclic operations on determinants. Read (Southwestern Section) Dec. 2, 1922. *American Mathematical Monthly*, vol. 30, No. 3, pp. 113–120; March–April, 1923.

Carmichael, R. D. Boundary value and expansion problems: oscillation, comparison, and expansion theorems. Read April 10, 1920. *American Journal of Mathematics*, vol. 44, No. 2, pp. 129–152; April, 1922.

Carver, W. B. Systems of linear inequalities. Read Dec. 29, 1920. *Annals of Mathematics*, (2), vol. 23, No. 3, pp. 212–220; March, 1922.

Chittenden, E. W. On a theorem in general analysis and the interrelations of eight fundamental properties of classes of functions. Read April 21, 1916. *American Journal of Mathematics*, vol. 44, No. 2, pp. 153–162; April, 1922.

Coble, A. B. Associated sets of points. Read Dec. 29, 1922. *Transactions of this Society*, vol. 24, No. 1, pp. 1–20; July, 1922.

—— Geometric aspects of the abelian modular functions of genus four (III). Read Dec. 31, 1919. *Proceedings of the National Academy of Sciences*, vol. 9, No. 6, pp. 183–187; June, 1923.

Coolidge, J. L. The Gaussian law of error for any number of variables. Read Dec. 27, 1922. *Transactions of this Society*, vol. 24, No. 2, pp. 135–143; Sept., 1922.

Craig, C. F. On the Riemann zeta function. Read Dec. 27, 1922. *This Bulletin*, vol. 29, No. 8, pp. 337–340; Oct., 1923.

Crum, W. L. The use of the median in determining seasonal variation. Read Oct. 28, 1922. *Journal of the American Statistical Association*, vol. 18, No. 141, pp. 607–614; March, 1923.

—— The resemblance between the ordinate of the periodogram and the correlation coëfficient. Read April 28, 1923. *Journal of the American Statistical Association*, vol. 18, No. 143, pp. 889–899; Sept., 1923.

—— Note on the reliability of a test, with special reference to the examinations set by the College Entrance Board. Read Oct. 28, 1922. *American Mathematical Monthly*, vol. 30, No. 6, pp. 296–301; Sept.–Oct., 1923.

Curtiss, D. R. A note on the preceding paper (by J. L. Walsh). Read April 14, 1922. *Transactions of this Society*, vol. 24, No. 3, pp. 181–184; Oct., 1922.

—— On Kellogg's diophantine problem. Read Dec. 28, 1921. *American Mathematical Monthly*, vol. 29, No. 10, pp. 380–387; Nov.–Dec., 1922.

—— Relations between kindred Riemannian P and Q functions. Read April 14, 1922. *This Bulletin*, vol. 29, No. 4, pp. 154–160; April, 1923.

Daus, P. H. Normal ternary continued fraction expansions for the cube roots of integers. Read (San Francisco) April 9, 1921. *American Journal of Mathematics*, vol. 44, No. 4, pp. 279–296; Oct., 1922.

Dickson, L. E. Algebras and their arithmetics. Read April 17, 1923. Chicago, University of Chicago Press, 1923.

—— Integral solutions of $x^2 - my^2 = zw$. Read (Southwestern Section) Dec. 1, 1923. *This Bulletin*, vol. 29, No. 10, pp. 464–467; Dec., 1923.

Dines, L. L. A primary classification of projective transformations in function space. Read Sept. 3, 1919. *American Journal of Mathematics*, vol. 44, No. 2, pp. 87–101; April, 1922.

Douglas, J. On certain two-point properties of general families of curves. Read April 28, 1917. *Transactions of this Society*, vol. 22, No. 3, pp. 289–310; July, 1921.

—— Determination of all systems of ∞^4 curves in space in which the sum of the angles of every triangle is two right angles. Read April 28, 1923. *This Bulletin*, vol. 29, No. 8, pp. 356–366; Oct., 1923.

DRESDEN, A. On the second derivatives of an extremal-integral with an application to a problem with variable end points. (Supplementary paper.) Read Dec. 28, 1914. *Transactions of this Society*, vol. 25, No. 2, pp. 190–192; April, 1923.

DUSHMAN, S. Some recent applications of the quantum theory to spectral series. Read Dec. 29, 1921. *Journal of the Optical Society of America*, vol. 6, No. 3, pp. 235–250; May, 1922.

EDINGTON, W. E. Abstract group definitions and applications. Read Dec. 29, 1922. *Transactions of this Society*, vol. 25, No. 2, pp. 193–210; April, 1923.

EISENHART, L. P. Orthogonal systems of hypersurfaces in a general Riemann space. Read April 28, 1923. *Transactions of this Society*, vol. 25, No. 2, pp. 259–280; April, 1923.

—— Symmetric tensors of the second order whose first covariant derivatives are zero. Read April 28, 1923. *Transactions of this Society*, vol. 25, No. 2, pp. 297–306; April, 1923.

EMCH, A. On plane algebraic curves which are invariant under a quadratic transformation. Read Dec. 28, 1921. *Tôhoku Mathematical Journal*, vol. 21, Nos. 3–4, pp. 310–326; Oct., 1922.

ETTLINGER, H. J. Cauchy's paper of 1814 on definite integrals. Read Sept. 2, 1919. *Annals of Mathematics*, (2), vol. 23, No. 3, pp. 255–270; March, 1922.

—— An elementary proof of a fundamental lemma concerning the limit of a sum. Read Sept. 7, 1922. *This Bulletin*, vol. 29, No. 5, pp. 219–223; May, 1923.

EVANS, G. C. Fundamental points of potential theory. Read Sept. 4, 1919, and Dec. 30, 1920. *The Rice Institute Pamphlet*, vol. 7, No. 4, pp. 252–329; Oct., 1920.

—— A simple theory of competition. Read Sept. 7, 1922. *American Mathematical Monthly*, vol. 29, No. 10, pp. 371–380; Nov.–Dec., 1922.

—— A Bohr-Langmuir transformation. Read Sept. 7, 1922. *Proceedings of the National Academy of Sciences*, vol. 9, No. 7, pp. 230–236; July, 1923.

EVERETT, H. S. Determination of all general homogeneous polynomials expressible as determinants whose elements are homogeneous polynomials. Read Dec. 28, 1921. *Transactions of this Society*, vol. 24, No. 3, pp. 185–194; Oct., 1922.

FELDSTEIN, M. M. Invariants of the linear group modulo p^k. Read April 13, 1923. *Transactions of this Society*, vol. 25, No. 2, pp. 223–238; April, 1923.

FISCHER, C. A. The kernel of the Stieltjes integral corresponding to a completely continuous transformation. Read April 23, 1921. *American Journal of Mathematics*, vol. 44, No. 4, pp. 237–246; Oct., 1922.

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BULLETIN

OF THE

AMERICAN
MATHEMATICAL SOCIETY

A HISTORICAL AND CRITICAL REVIEW
OF MATHEMATICAL SCIENCE

EDITED BY

EARLE RAYMOND HEDRICK
WALLIE ABRAHAM HURWITZ JOHN WESLEY YOUNG

WITH THE ASSISTANCE OF

ROBERT WILBUR BURGESS DERRICK NORMAN LEHMER
DUNHAM JACKSON TULLIO LEVI-CIVITA
EDWARD KASNER HENRY LEWIS RIETZ

VOLUME XXIX, NUMBER 10

DECEMBER, 1923

PUBLISHED BY THE SOCIETY
LANCASTER, PA., AND NEW YORK

1923

Entered at the Post Office at Lancaster, Pa., as second class matter
PUBLISHED MONTHLY, EXCEPT AUGUST AND SEPTEMBER

Whole No. 319 $7.00 a Year

BULLETIN

OF THE

American Mathematical Society

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THE EDITORS

Dec. 1, 1923

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EDITED BY
HENRI VILLAT
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This Journal, which has been among the foremost exponents of mathematics for nearly a century, was in danger of discontinuance owing to the changed financial situation caused by the war. By prompt action of a group of French mathematicians, its publication has been assured for the immediate future. This group contains, in addition to the editors, such eminent men as Appell, Borel, Boussinesq, Brillouin, Cartan, Drach, Goursat, Guichard, Hadamard, Koenigs, Lebesgue, Montel, Painlevé, Vessiot.

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which is a review of mathematical literature that has been a standard source of reference for many years. See this Bulletin, pp. 284, 333, (1923), and p. iv of the October issue. Books for review, etc., should be sent to the editor, L. Lichtenstein, Cunostr. 65, Berlin-Grunewald. Subscriptions should be sent to

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OFFICIAL COMMUNICATIONS

Meetings of the Society have been fixed at the following times and places:

THE SAN FRANCISCO SECTION, AT THE UNIVERSITY OF WASHINGTON, IN SEATTLE, December 22, 1923. See also this Bulletin, vol. 29, No. 9 (Nov., 1923), p. 425.

Abstracts must be in the hands of the acting Secretary of the Section, D. N. Lehmer, Berkeley, Cal., not later than November 29.

THE ANNUAL MEETING OF THE SOCIETY, IN NEW YORK CITY, December 27–28, 1923. See also this Bulletin, vol. 29, No. 4 (April, 1923), p. 188.

Abstracts must be in the hands of the Secretary of the Society not later than November 29.

THE TWENTIETH WESTERN MEETING OF THE SOCIETY (Fifty-second Meeting of the Chicago Section), in Cincinnati, December 28–29, 1923. See also this Bulletin, vol. 29, No. 5 (May, 1923), p. 197; and No. 9 (Nov., 1923), p. 425.

Abstracts must be in the hands of the Secretary of the Chicago Section, Arnold Dresden, 2114 Vilas St., Madison, Wis., not later than November 29.

NEW YORK CITY, February 24, 1924.

Abstracts must be in the hands of the Secretary of the Society not later than February 10.

THE SAN FRANCISCO SECTION, AT STANFORD UNIVERSITY, April 5, 1924.

Abstracts should be in the hands of the acting Secretary of the Section, C. A. Noble, Berkeley, Cal., not later than March 22.

NEW YORK CITY, May 3, 1924.

Abstracts must be in the hands of the Secretary of the Society not later than April 19.

R. G. D. RICHARDSON,
Secretary of the Society.

COLLECTED WORKS OF LEONARD EULER

Libraries or individuals wishing for information with a view to joining in promoting this great international undertaking, should communicate with the Official Representative of the Euler Committee for the United States and Canada,

R. C. ARCHIBALD, Brown University, Providence, R. I.

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Any communication intended for this Bulletin may be addressed to the American Mathematical Society, Prince and Lemon Streets, Lancaster, Pa., or may be sent to separate officials of the Society as follows:

Articles for insertion in the Bulletin should be addressed to E. R. Hedrick, Editor of the Bulletin, 304 Hicks Ave., Columbia, Mo. Reviews should be sent to J. W. Young, Dartmouth College, Hanover, N. H. Notes should be sent to R. W. Burgess, Brown University, Providence, R. I.

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Changes of address of members, exchanges, and subscribers should be communicated at once to the Secretary of the Society, R. G. D. Richardson, 501 West 116th Street, New York.

The initiation fees ($5.00) and annual dues ($6.00) of members of the Society are payable to the Treasurer of the Society, W. B. Fite, 501 West 116th Street, New York.

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